ME 211 Statics and Strength of Materials

Chapter 12

Transformations of Stress and Strain

Introduction

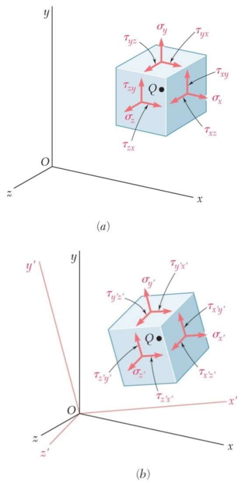


Fig. 7.1 General state of stress at a point: (a) referred to {xyz}, (b) referred to {x′y′z′}.

The most general state of stress at a point may be represented by 6 components,

$\sigma_x, \sigma_y, \sigma_z$ normal stresses

$\tau_{xy}, \tau_{yz}, \tau_{zx}$ shearing stresses

(Note : $\tau_{xy} = \tau_{yx}, \tau_{yz} = \tau_{zy}, \tau_{zx} = \tau_{xz}$)

Same state of stress is represented by a different set of components if axes are rotated.

The first part of the chapter is concerned with how the components of stress are transformed under a rotation of the coordinate axes. The second part of the chapter is devoted to a similar analysis of the transformation of the components of strain.

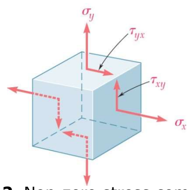


Fig. 7.2 Non-zero stress components for state of plane stress.

Plane Stress - state of stress in which two faces of the cubic element are free of stress. For the illustrated example, the state of stress is defined by

$$\sigma_x,\ \sigma_y,\ \tau_{xy} \quad \text{and} \quad \sigma_z = \tau_{zx} = \tau_{zy} = 0.$$

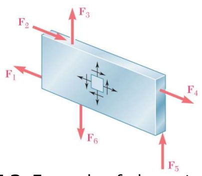


Fig. 7.3 Example of plane stress: thin plate subjected to only in-plane loads.

State of plane stress occurs in a thin plate subjected to forces acting in the midplane of the plate.

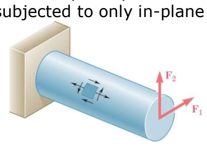


Fig. 7.4 Example of plane stress: free surface of a structural component.

State of plane stress also occurs on the free surface of a structural element or machine component, i.e., at any point of the surface not subjected to an external force.

Transformation of Plane Stress

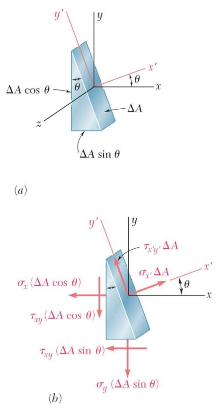


Fig. 7.6 Stress transformation equations are determined by considering an arbitrary prismatic wedge element. (a) Geometry of the element. (b) Free-body diagram.

Consider the conditions for equilibrium of a prismatic element with faces perpendicular to the x, y, and x' axes.

$$\sum F_{x'} = 0 = \sigma_{x'}\Delta A - \sigma_x(\Delta A\cos\theta)\cos\theta - \tau_{xy}(\Delta A\cos\theta)\sin\theta$$
$$-\sigma_y(\Delta A\sin\theta)\sin\theta - \tau_{xy}(\Delta A\sin\theta)\cos\theta$$

$$\sum F_{y'} = 0 = \tau_{x'y'}\Delta A + \sigma_x(\Delta A\cos\theta)\sin\theta - \tau_{xy}(\Delta A\cos\theta)\cos\theta$$
$$-\sigma_y(\Delta A\sin\theta)\cos\theta + \tau_{xy}(\Delta A\sin\theta)\sin\theta$$

The equations may be rewritten to yield

$$\sigma_{x'} = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{y'} = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x-\sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Transformation of Plane Stress

Principal Stresses

The transformation equations for two-dimensional stress indicate that the normal stress $\sigma_{x'}$ and shearing stress $\tau_{x'y'}$ vary continuously as the axes are rotated through the angle θ. Therefore there should be minimum and maximum values for the stress components for particular orientations of x′ - y′. We can find the orientation for those values by setting $d\sigma_{x'}/d\theta = 0$ from the previous page. By doing so, we have

$$-(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

Which gives

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Since tan 2θ = tan(π + 2θ), there are two values θ_p describing directions. These are the principal directions along which the maximum and minimum normal stresses (σ_1 and σ_2) act.

$$-(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

States that $\tau_{x'y'} = 0$ on a principal plane (check the transformation equation in the previous page). A principal plane is thus a plane of zero shear. When you substitute tan2θp equation into σx′ relation:

$$\sigma_{\max,\min} = \sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Principal Stresses

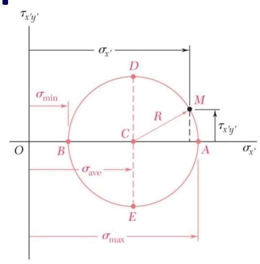


Fig. 7.7 Circular relationship of transformed stresses.

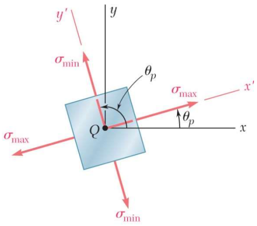


Fig. 7.9 Principal stresses.

The previous equations are combined to yield parametric equations for a circle,

$$(\sigma_{x'} - \sigma_{ave})^2 + \tau_{x'y'}^2 = R^2$$

where

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Principal stresses occur on the *principal planes of stress* with zero shearing stresses.

$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Note : defines two angles separated by 90^o

Maximum Shearing Stress

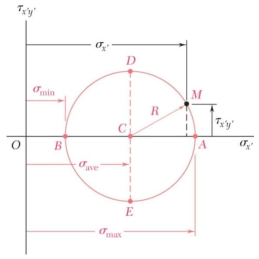


Fig. 7.7 Circular relationship of transformed stresses.

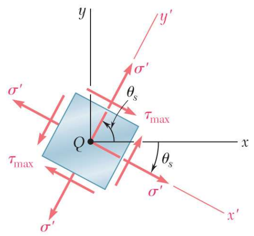


Fig. 7.10 Maximum shearing stress.

Maximum shearing stress occurs for $\sigma_{x'} = \sigma_{ave}$

$$\tau_{max} = R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

Note : defines two angles separated by 90° and offset from θ_p by 45°

$$\sigma' = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$$

Concept Application 7.1

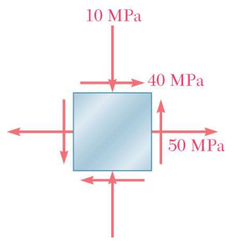


Fig. 7.11a Plane stress element.

For the state of plane stress shown, determine (a) the principal planes, (b) the principal stresses, (c) the maximum shearing stress and the corresponding normal stress.

SOLUTION:

Find the element orientation for the principal stresses from

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Determine the principal stresses from

$$\sigma_{max,min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Calculate the maximum shearing stress with

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma' = \frac{\sigma_x + \sigma_y}{2}$$

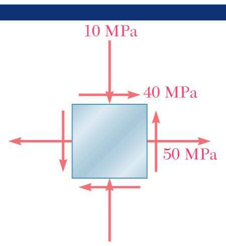


Fig. 7.11a Plane stress element.

$\sigma_x = +50\,\text{MPa}$ $\quad \tau_{xy} = +40\,\text{MPa}$

$\sigma_x = -10\,\text{MPa}$

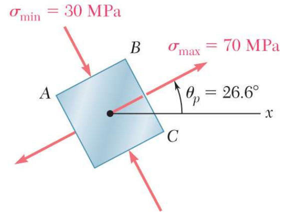


Fig. 7.11b Plane stress element oriented in principal directions.

SOLUTION:

Find the element orientation for the principal stresses from

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(+40)}{50-(-10)} = 1.333$$

$$2\theta_p = 53.1°,\ 233.1°$$

$$\boxed{\theta_p = 26.6°,\ 116.6°}$$

Determine the principal stresses from

$$\sigma_{max,min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= 20 \pm \sqrt{(30)^2 + (40)^2}$$

$$\boxed{\begin{array}{l}\sigma_{max} = 70\,\text{MPa}\\ \sigma_{min} = -30\,\text{MPa}\end{array}}$$

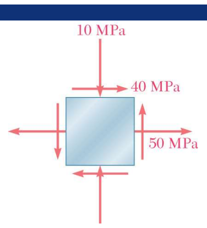


Fig. 7.11a Plane stress element.

$\sigma_x = +50\,\text{MPa}$ $\quad \tau_{xy} = +40\,\text{MPa}$

$\sigma_x = -10\,\text{MPa}$

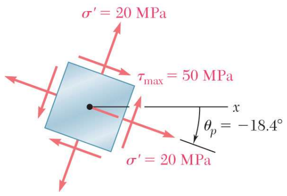


Fig. 7.11d Plane stress element showing maximum shear orientation.

Calculate the maximum shearing stress with

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \sqrt{(30)^2 + (40)^2}$$

$$\boxed{\tau_{max} = 50\,\text{MPa}}$$

$$\theta_s = \theta_p - 45$$

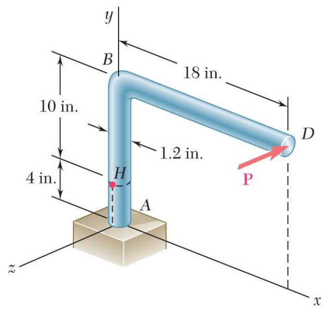
$$\boxed{\theta_s = -18.4°,\ 71.6°}$$

The corresponding normal stress is

$$\sigma' = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{50-10}{2}$$

$$\boxed{\sigma' = 20\,\text{MPa}}$$

Sample Problem 7.1



A single horizontal force *P* of with a magnitude of 150 lb is applied to end D of lever *ABD*. Determine (a) the normal and shearing stresses on an element at point *H* having sides parallel to the *x* and *y* axes, (b) the principal planes and principal stresses at the point *H*.

SOLUTION:

Determine an equivalent force-couple system at the center of the transverse section passing through *H*.

Evaluate the normal and shearing stresses at *H*.

Determine the principal planes and calculate the principal stresses.

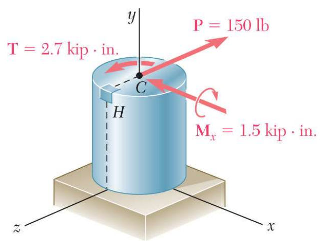


Fig. 1 Equivalent force-couple system acting on transverse section containing point *H*.

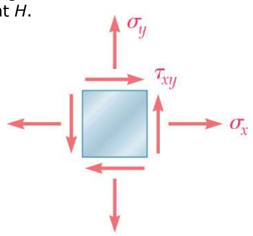


Fig. 2 General plane stress element (showing positive directions).

SOLUTION:

Determine an equivalent force-couple system at the center of the transverse section passing through *H*.

$$P = 150\,\text{lb}$$
$$T = (150\,\text{lb})(18\,\text{in}) = 2.7\,\text{kip}\cdot\text{in}$$
$$M_x = (150\,\text{lb})(10\,\text{in}) = 1.5\,\text{kip}\cdot\text{in}$$

Evaluate the normal and shearing stresses at *H*.

$$\sigma_y = +\frac{Mc}{I} = +\frac{(1.5\,\text{kip}\cdot\text{in})(0.6\,\text{in})}{\frac{1}{4}\pi(0.6\,\text{in})^4}$$

$$\tau_{xy} = +\frac{Tc}{J} = +\frac{(2.7\,\text{kip}\cdot\text{in})(0.6\,\text{in})}{\frac{1}{2}\pi(0.6\,\text{in})^4}$$

$$\sigma_x = 0 \quad \sigma_y = +8.84\,\text{ksi} \quad \tau_{xy} = +7.96\,\text{ksi}$$

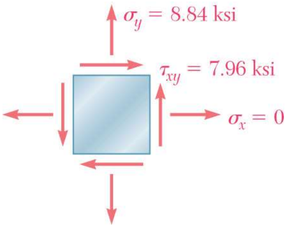


Fig. 3 Stress element at point *H*.

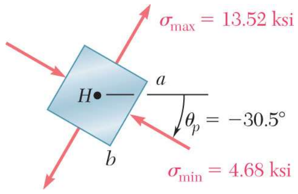


Fig. 4 Stress element at point *H* oriented in principal directions.

Determine the principal planes and calculate the principal stresses.

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(7.96)}{0 - 8.84} = -1.8$$

$$2\theta_p = -61.0°, 119°$$

$$\boxed{\theta_p = -30.5°,\ 59.5°}$$

$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{0 + 8.84}{2} \pm \sqrt{\left(\frac{0 - 8.84}{2}\right)^2 + (7.96)^2}$$

$$\boxed{\begin{aligned}\sigma_{\max} &= +13.52\,\text{ksi}\\ \sigma_{\min} &= -4.68\,\text{ksi}\end{aligned}}$$

Mohr's Circle for Plane Stress

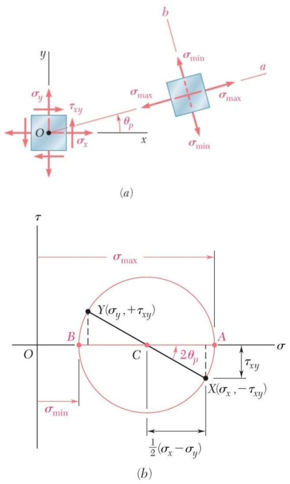


Fig. 7.12 (a) Plane stress element and the orientation of principal planes. (b) corresponding Mohr's circle.

With the physical significance of Mohr's circle for plane stress established, it may be applied with simple geometric considerations. Critical values are estimated graphically or calculated.

For a known state of plane stress $\sigma_x, \sigma_y, \tau_{xy}$ plot the points *X* and *Y* and construct the circle centered at *C*.

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

The principal stresses are obtained at *A* and *B*.

$$\sigma_{\max,\min} = \sigma_{ave} \pm R$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

The direction of rotation of *Ox* to *Oa* is the same as *CX* to *CA*.

With Mohr's circle uniquely defined, the state of stress at other axes orientations may be depicted.

For the state of stress at an angle θ with respect to the *xy* axes, construct a new diameter *X'Y'* at an angle 2θ with respect to *XY*.

Normal and shear stresses are obtained from the coordinates *X'Y'*.

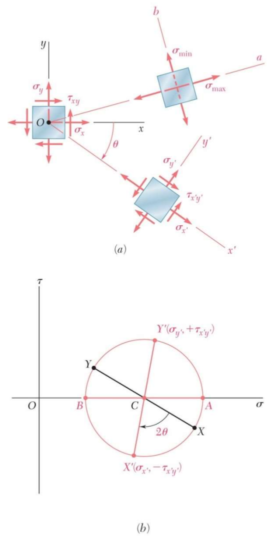
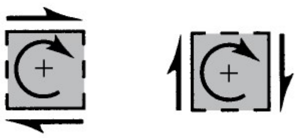


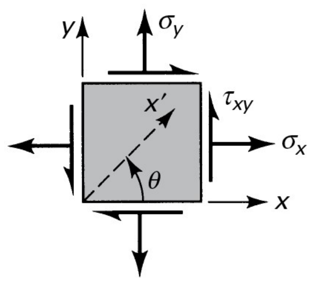
Fig. 7.13 (a) Stress element referenced to *xy* axes, transformed to obtain components referenced to *x'y'* axes. (b) Corresponding Mohr's circle.

Generation of Mohr's Circle for Two-Dimensional Stress

A graphical technique permits the rapid transformation of stress from one plane to another and leads to the determination of the maximum normal and shear stresses. In this approach, trandformation equations **are depicted by a stress circle**, called **Mohr's circle**. In the Mohr representation, the normal stresses obey the sign convention described before. However, for the purposes of constructing and reading values of stress from Mohr's circle, the sign convention for shear stress is as follows:

If the shearing stresses on opposite faces of an element would produce shearing forces that result in a clockwise couple, as shown below, these stresses are regarded as positive. Accordingly, the shearing stresses on the y faces of the element below are taken as positive (as before), but those on the x faces are now negative.





Given σ_x, σ_y, and τ_{xy} with algebraic signs in accordance with this sign convention, the procedure for obtaining Mohr's circle is as follows:

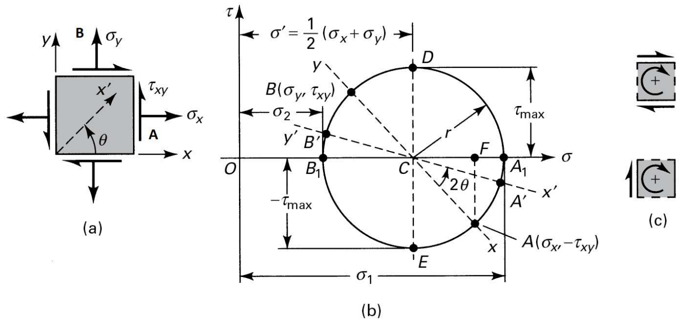


Figure 1.15. (a) Stress element; (b) Mohr's circle of stress; (c) interpretation of positive shearing stresses.

- Establish a rectangular coordinate system, indicating $+\tau$ and $+\sigma$.
- Locate the center C of the circle on the horizontal axis at a distance ½ $(\sigma_x + \sigma_y)$ from the origin.
- Locate point *A* by coordinates σ_x and $-\tau_{xy}$. We have placed A on +x face and B on +y face.
- Draw a circle with center at C and of radius equal to CA.
- Draw line AB through C.
- The angles on the circle are measured in the same direction as θ is measured in Fig. 1.15a. An angle of 2θ on the circle corresponds to an angle of θ on the element.

The state of stress associated with the original x and y planes corresponds to points A and B on the circle, respectively. Points lying on diameters other than AB, such as A′ and B′, define states of stress with respect to any other set of x′ and y′ planes rotated relative to the original set through an angle θ

Points A_1 and B_1 on the circle locate the principal stresses and provide their magnitudes as defined by equations, while points D and E represent the maximum shearing stresses. The radius of the circle is

$$CA = \sqrt{CF^2 + AF^2} \qquad \textbf{(a)}$$

Where

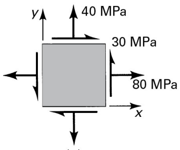
$$CF = \tfrac{1}{2}(\sigma_x - \sigma_y), \qquad AF = \tau_{xy}$$

Thus, the radius equals the magnitude of the maximum shearing stress. Mohr's circle shows that the planes of maximum shear are always located at 45° from planes of principal stress.

The use of Mohr's circle is illustrated in the next example.

Mohr's Circle Example

At a point in the structural member, the stresses are represented as

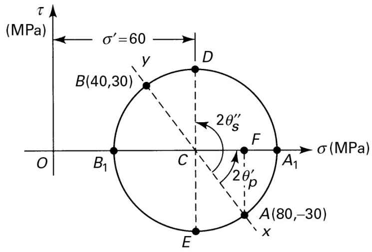


(a)

Find by employing Mohr's circle:

(a) the magnitude and orientation of the principal stresses
(b) the magnitude and orientation of the maximum shearing stresses and associated normal stresses.
(c) In each case, show the results on a properly oriented element; represent the stress tensor in matrix form.

The center of the circle is at (40 + 80)/2 = 60 MPa on the σ axis. Stress state at +x face or point A = (+80, -30) and stress state on +y face or point B = (+40, +30). Connect CA or radius or AB for diameter and draw the Mohr's circle.



(b)

σ_1 = 60 + sqrt(30^2+$(80-60)^2$) = 60 + **36.05** = 96.05MPa
σ_2 = 60 - sqrt(30^2+$(80-60)^2$) = 60 - **36.05** = 23.95MPa

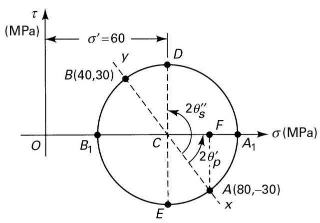
Θ_{p1}= 56.31/2 = **28.15°**
Θ_{p2} = (180 + 56.31)/2 = 118.16°

Mohr's circle clearly indicates that Θ_{p1} locates the σ_1 plane

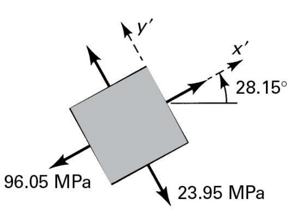
The maximum shearing stresses are given by points D and E.

As calculated in the previous page R = 36.05MPa
Therefore τ_{max} = 36.05MPa

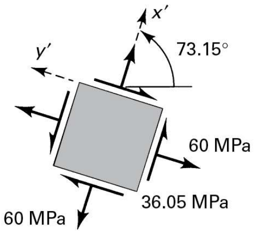
$2\Theta_{s1}$= $2\Theta_{p1}$ + 90 = 146.31 → Θ_{s1} = **73.15°**
$2\Theta_{s2}$ = 146.31 + 180 = 326.31 → Θ_{s1} = 163.15°



(b)



(c)



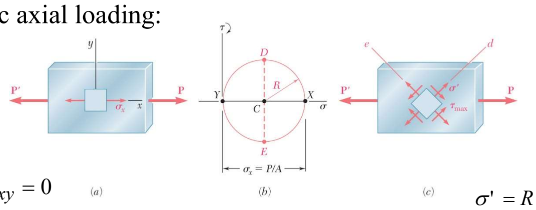
(d)

We may now describe the state of stress at the point in the following matrix forms:

$$\begin{bmatrix} 80 & 30 \\ 30 & 40 \end{bmatrix}, \begin{bmatrix} 96.05 & 0 \\ 0 & 23.95 \end{bmatrix}, \begin{bmatrix} 60 & -36.06 \\ -36.06 & 60 \end{bmatrix}$$

Mohr’s Circle for Plane Stress

Mohr’s circle for centric axial loading:

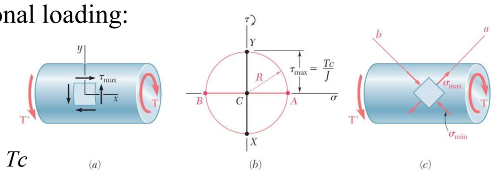


$$\sigma_x = \frac{P}{A}, \quad \sigma_y = \tau_{xy} = 0$$

$$\sigma' = R = \tau_{max} = \frac{P}{2A}$$

Fig. 7.17 (a) Member under centric axial load. (b) Mohr’s circle. (c) Element showing planes of maximum shearing stress.

Mohr’s circle for torsional loading:



$$\sigma_x = \sigma_y = 0 \quad \tau_{xy} = \frac{Tc}{J}$$

$$\sigma_{max,\,min} = \pm R = \pm\frac{Tc}{J}$$

Fig. 7.18 (a) Member under torsional load. (b) Mohr’s circle. (c) Element showing orientation of principal stresses.

Concept Application 7.2

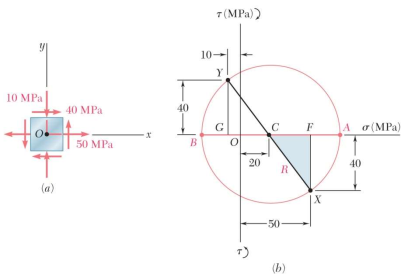


Fig. 7.16 (a) Plane stress element.
(b) Corresponding Mohr's circle.

For the state of plane stress considered in Concept Application 7.1, (a) construct Mohr’s circle, (b) determine the principal stresses, (c) determine the maximum shearing stress and the corresponding normal stress.

SOLUTION:

Construction of Mohr’s circle

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{(50) + (-10)}{2} = 20\,\text{MPa}$$

$$CF = 50 - 20 = 30\,\text{MPa} \qquad FX = 40\,\text{MPa}$$

$$R = CX = \sqrt{(30)^2 + (40)^2} = 50\,\text{MPa}$$

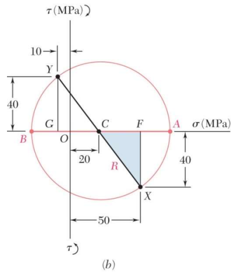


Fig. 7.16 (b) Mohr's circle showing face X and Y.

Principal planes and stresses

$$\sigma_{max} = OA = OC + CA = 20 + 50$$

$$\boxed{\sigma_{max} = 70\,\text{MPa}}$$

$$\sigma_{min} = OB = OC - BC = 20 - 50$$

$$\boxed{\sigma_{min} = -30\,\text{MPa}}$$

$$\tan 2\theta_p = \frac{FX}{CP} = \frac{40}{30}$$

$$2\theta_p = 53.1°$$

$$\boxed{\theta_p = 26.6°}$$

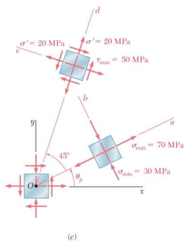
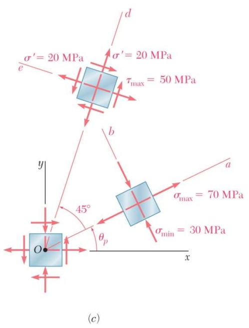


Fig. 7.16 (c) Stress element orientations for principal and maximum shear orientations.



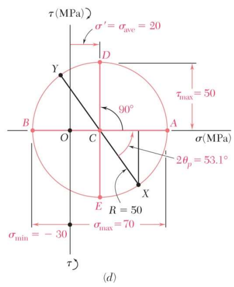


Fig. 7.16 (c) Stress element orientations for principal and maximum shear orientations.

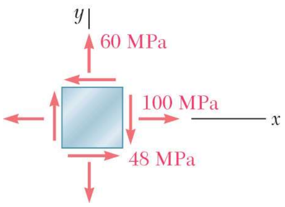
Fig. 7.16 (d) Mohr's circle used to determine principal and maximum shearing stresses.

Maximum shear stress

$$\theta_s = \theta_p + 45° \qquad \tau_{max} = R \qquad \sigma' = \sigma_{ave}$$

$$\boxed{\theta_s = 71.6°} \qquad \boxed{\tau_{max} = 50\,\text{MPa}} \qquad \boxed{\sigma' = 20\,\text{MPa}}$$

Sample Problem 7.2



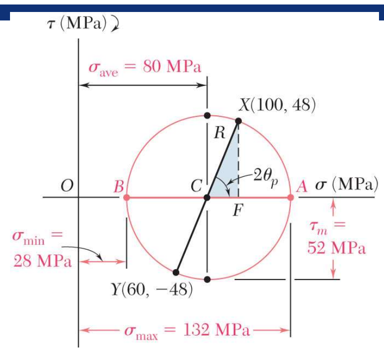


Fig. 1 Mohr's circle for given stress state.

For the state of stress shown, determine (a) the principal planes and the principal stresses, and (b) the stress components exerted on the element obtained by rotating the given element counterclockwise through 30º.

SOLUTION:
Construct Mohr's circle

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{100+60}{2} = 80\,\text{MPa}$$

$$R = \sqrt{(CF)^2 + (FX)^2} = \sqrt{(20)^2 + (48)^2} = 52\,\text{MPa}$$

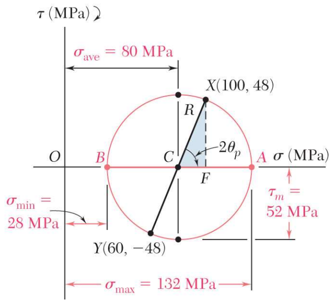


Fig. 1 Mohr's circle for given stress state.

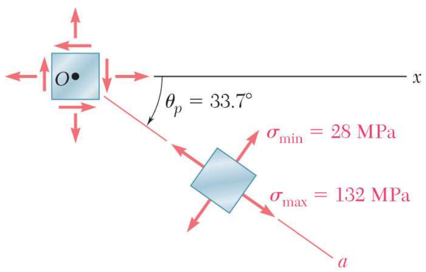


Fig. 2 Orientation of principal stress element.

Principal planes and stresses

$$\tan 2\theta_p = \frac{XF}{CF} = \frac{48}{20} = 2.4$$

$$2\theta_p = 67.4°$$

$$\boxed{\theta_p = 33.7° \text{ clockwise}}$$

$$\sigma_{max} = OA = OC + CA = 80 + 52$$

$$\boxed{\sigma_{max} = +132\,\text{MPa}}$$

$$\sigma_{max} = OA = OC - BC = 80 - 52$$

$$\boxed{\sigma_{min} = +28\,\text{MPa}}$$

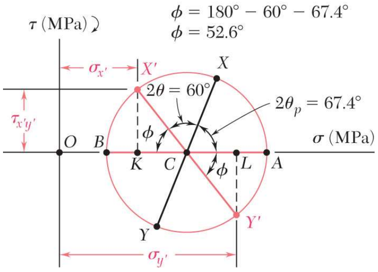


Fig. 3 Mohr's circle analysis for element rotation of 30° counterclockwise.

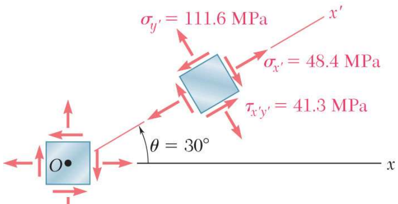


Fig. 4 Stress components obtained by rotating original element 30° counterclockwise.

Stress components after rotation by 30°

Points X' and Y' on Mohr's circle that correspond to stress components on the rotated element are obtained by rotating XY counterclockwise through $2\theta = 60°$

$$\phi = 180° - 60° - 67.4° = 52.6°$$

$$\sigma_{x'} = OK = OC - KC = 80 - 52\cos 52.6°$$

$$\sigma_{y'} = OL = OC + CL = 80 + 52\cos 52.6°$$

$$\tau_{x'y'} = KX' = 52\sin 52.6°$$

$$\sigma_{x'} = +48.4\,\text{MPa}$$
$$\sigma_{y'} = +111.6\,\text{MPa}$$
$$\tau_{x'y'} = 41.3\,\text{MPa}$$

General State of Stress

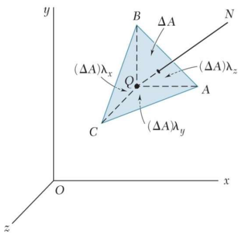


Fig. 7.19 Stress tetrahedron at point Q with three faces parallel to the coordinate planes.

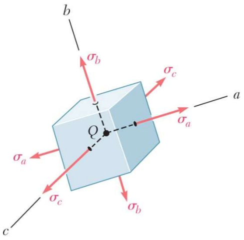


Fig. 7.21 General stress element oriented to principal axes.

Consider the general 3D state of stress at a point and the transformation of stress from element rotation

State of stress at Q defined by: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$

Consider tetrahedron with face perpendicular to the line QN with direction cosines: $\lambda_x, \lambda_y, \lambda_z$

The requirement $\sum F_n = 0$ leads to,

$$\sigma_n = \sigma_x\lambda_x^2 + \sigma_y\lambda_y^2 + \sigma_z\lambda_z^2 + 2\tau_{xy}\lambda_x\lambda_y + 2\tau_{yz}\lambda_y\lambda_z + 2\tau_{zx}\lambda_z\lambda_x$$

Form of equation guarantees that an element orientation can be found such that

$$\sigma_n = \sigma_a\lambda_a^2 + \sigma_b\lambda_b^2 + \sigma_c\lambda_c^2$$

These are the principal axes and principal planes and the normal stresses are the principal stresses.

Stresses in Thin-Walled Pressure Vessels

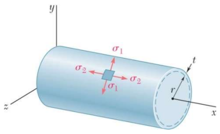


Fig. 7.40 Pressurized cylindrical vessel.

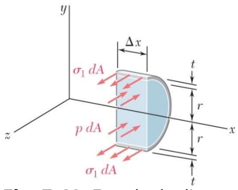


Fig. 7.41 Free-body diagram to determine hoop stress in a cylindrical pressure vessel.

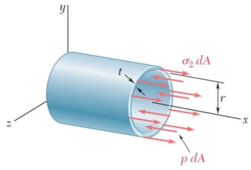


Fig. 7.42 Free-body diagram to determine longitudinal stress.

Cylindrical vessel with principal stresses

σ_1 = hoop stress

σ_2 = longitudinal stress

Hoop stress:

$$\sum F_z = 0 = \sigma_1(2t\,\Delta x) - p(2r\,\Delta x)$$

$$\sigma_1 = \frac{pr}{t}$$

Longitudinal stress:

$$\sum F_x = 0 = \sigma_2(2\pi r t) - p(\pi r^2)$$

$$\sigma_2 = \frac{pr}{2t}$$

$$\sigma_1 = 2\sigma_2$$

Stresses in Thin-Walled Pressure Vessels

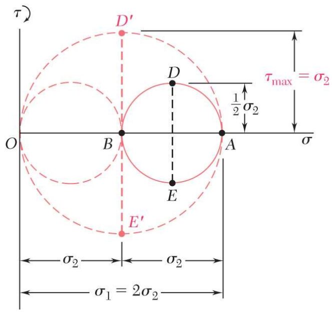


Fig. 7.43 Mohr's circle for element of cylindrical pressure vessel.

Points A and B correspond to hoop stress, σ_1, and longitudinal stress, σ_2

Maximum in-plane shearing stress:

$$\tau_{\max(\text{in-plane})} = \frac{1}{2}\sigma_2 = \frac{pr}{4t}$$

Maximum out-of-plane shearing stress corresponds to a 45° rotation of the plane stress element around a longitudinal axis

$$\tau_{\max} = \sigma_2 = \frac{pr}{2t}$$

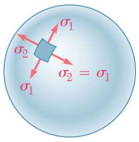


Fig. 7.44 Pressurized spherical vessel.

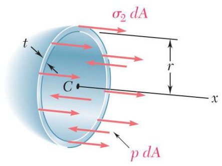


Fig. 7.45 Free-body diagram to determine spherical pressure vessel stress.

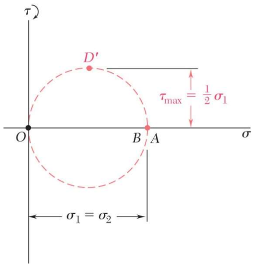


Fig. 7.46 Mohr's circle for element of spherical pressure vessel.

Spherical pressure vessel:

$$\sigma_1 = \sigma_2 = \frac{pr}{2t}$$

Mohr's circle for in-plane transformations reduces to a point

$$\sigma = \sigma_1 = \sigma_2 = \text{constant}$$

$$\tau_{\max(\text{in-plane})} = 0$$

Maximum out-of-plane shearing stress

$$\tau_{\max} = \tfrac{1}{2}\sigma_1 = \frac{pr}{4t}$$

Transformation of Plane Strain

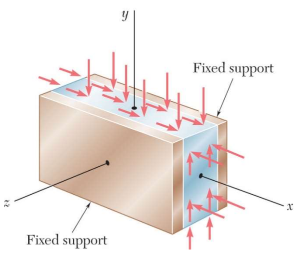


Fig. 7.47 Plane strain example: laterally restrained by fixed supports.

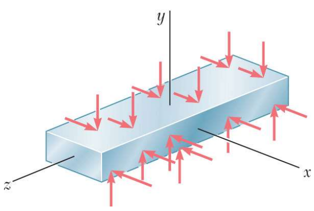


Fig. 7.48 Plane strain example: bar of infinite length in z direction.

Plane strain - deformations of the material take place in parallel planes and are the same in each of those planes.

Plane strain occurs in a plate subjected along its edges to a uniformly distributed load and restrained from expanding or contracting laterally by smooth, rigid and fixed supports

components of strain :

$$\varepsilon_x \; \varepsilon_y \; \gamma_{xy} \quad \left(\varepsilon_z = \gamma_{zx} = \gamma_{zy} = 0\right)$$

Example: Consider a long bar subjected to uniformly distributed transverse loads. State of plane stress exists in any transverse section not located too close to the ends of the bar.

Transformation of Plane Strain

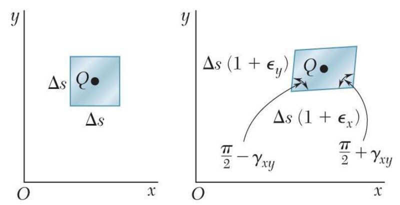


Fig. 7.49 Plane strain element: undeformed and deformed.

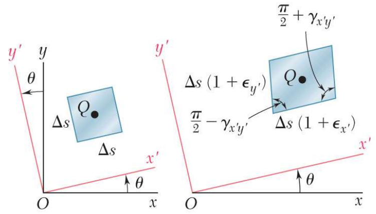


Fig. 7.50 Transformation of plane strain element in undeformed and deformed orientations.

State of strain at the point Q results in different strain components with respect to the xy and $x'y'$ reference frames.

$$\varepsilon(\theta) = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy}\sin\theta\cos\theta$$

$$\varepsilon_{OB} = \varepsilon(45°) = \tfrac{1}{2}\left(\varepsilon_x + \varepsilon_y + \gamma_{xy}\right)$$

$$\gamma_{xy} = 2\varepsilon_{OB} - \left(\varepsilon_x + \varepsilon_y\right)$$

Applying the trigonometric relations used for the transformation of stress,

$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\frac{\gamma_{x'y'}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

Mohr's Circle for Plane Strain

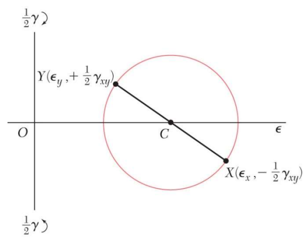


Fig. 7.53 Mohr's circle for plane strain.

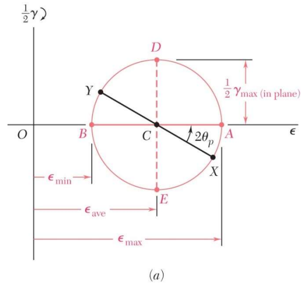


Fig. 7.54a Mohr's circle for plane strain, showing principal strains and maximum in-plane shearing strain.

The equations for the transformation of plane strain are of the same form as the equations for the transformation of plane stress - *Mohr's circle techniques apply.*

Abscissa for the center C and radius R ,

$$\varepsilon_{ave} = \frac{\varepsilon_x + \varepsilon_y}{2} \qquad R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

Principal axes of strain and principal strains,

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

$$\varepsilon_{max} = \varepsilon_{ave} + R \qquad \varepsilon_{min} = \varepsilon_{ave} - R$$

Maximum in-plane shearing strain,

$$\gamma_{max} = 2R = \sqrt{\left(\varepsilon_x - \varepsilon_y\right)^2 + \gamma_{xy}^2}$$

Measurements of Strain: Strain Rosette

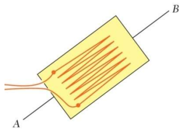


Fig. 7.63 Electrical strain gage.

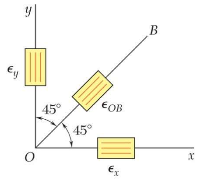


Fig. 7.64 Strain rosette that measures normal strains in direction of x, y, and bisector OB.

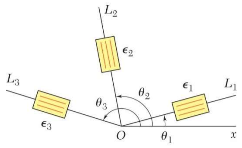


Fig. 7.65 Generalized strain gage rosette arrangement.

Strain gages indicate normal strain through changes in resistance.

With a 45° rosette, ε_x and ε_y are measured directly. γ_{xy} is obtained indirectly with,

$$\gamma_{xy} = 2\varepsilon_{OB} - \left(\varepsilon_x + \varepsilon_y\right)$$

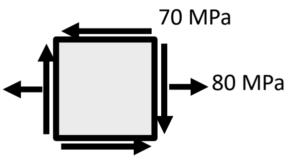
Normal and shearing strains may be obtained from normal strains in any three directions,

$$\varepsilon_1 = \varepsilon_x \cos^2\theta_1 + \varepsilon_y \sin^2\theta_1 + \gamma_{xy} \sin\theta_1 \cos\theta_1$$

$$\varepsilon_2 = \varepsilon_x \cos^2\theta_2 + \varepsilon_y \sin^2\theta_2 + \gamma_{xy} \sin\theta_2 \cos\theta_2$$

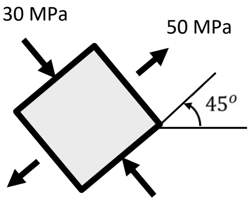
$$\varepsilon_3 = \varepsilon_x \cos^2\theta_3 + \varepsilon_y \sin^2\theta_3 + \gamma_{xy} \sin\theta_3 \cos\theta_3$$

Example 1:



(i)

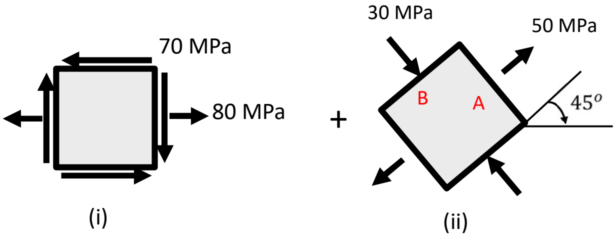
+



(ii)

Determine the principal planes and principal stresses for the state of plane stress resulting from the superposition of the two states of plane stress shown.

- Rotate element (ii) by 45^o clockwise



Tension(+) Compression(-)

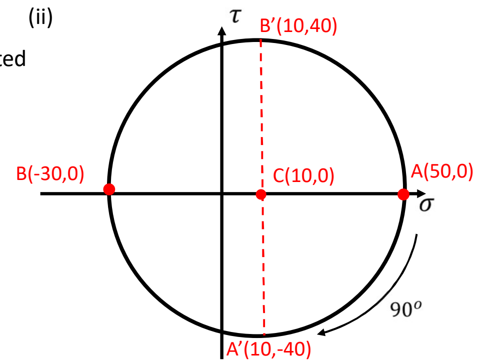
A(50,0), B(-30,0)

Shear 0 Shear 0

- Center and radius of the circle are calculated

$$C = \frac{-30 + 50}{2} = 10 \qquad C \rightarrow (10,0)$$

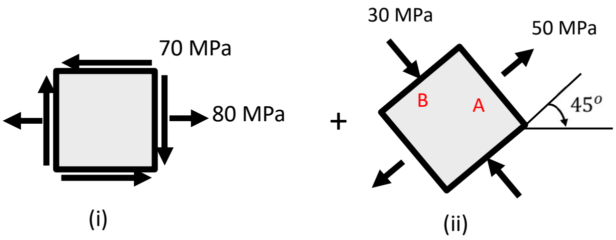
$$R = (50 - 10) = 40$$



- Rotate element (ii) by 45^o clockwise

45^o on stress state, 90^o on Mohr's circle

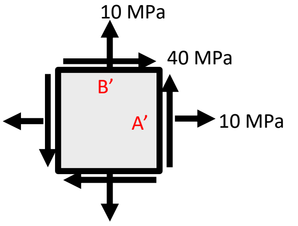
After 90^o rotation we get: A'(10,-40), B'(10,40)



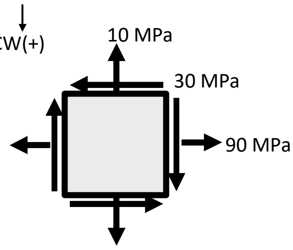
After 45^o rotation of stress state we get: A'(10,-40), B'(10,40)

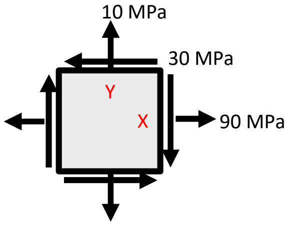
CCW(-) CW(+)

- Element (ii) becomes after 45 rotation



Summing (i) and new (ii) we get





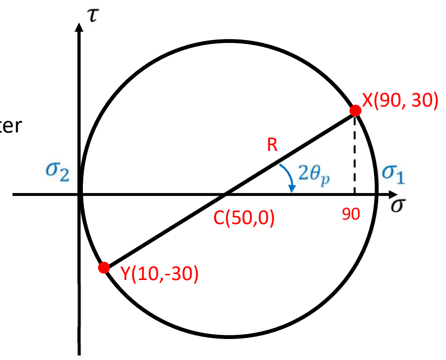
Tension(+) Tension(+)

X(90,30), Y(10,-30)

CW(+) CCW(-)

$$C = \frac{90 + 10}{2} = 50$$

$$R = \sqrt{(90-50)^2 + 30^2} = 50$$



- Find principal stresses and planes
- Draw Mohr's circle for new stress state
- Principal stresses are calculated using center and radius values

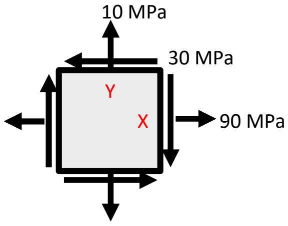
$\sigma_1 = C + R = 50 + 50 = 100\ MPa$

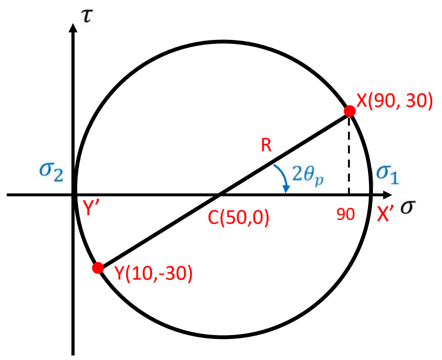
$\sigma_2 = C - R = 50 - 50 = 0\ MPa$

- Find principal planes

$\tan(2\theta_p) = \frac{30}{90-50} \rightarrow 2\theta_p = 36.87^o$

$\theta_p = 18.4^o$(CW) or 71.6^o(CCW)





$\theta_p = 18.4^o$(CW) or 71.6^o(CCW)

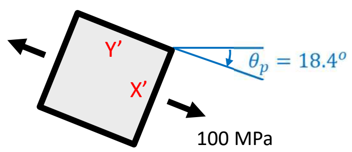
- XY is the state after the superposition
- We rotate this state 18.4 degrees CW to get the principal state

$\sigma_1 = 100\ MPa$

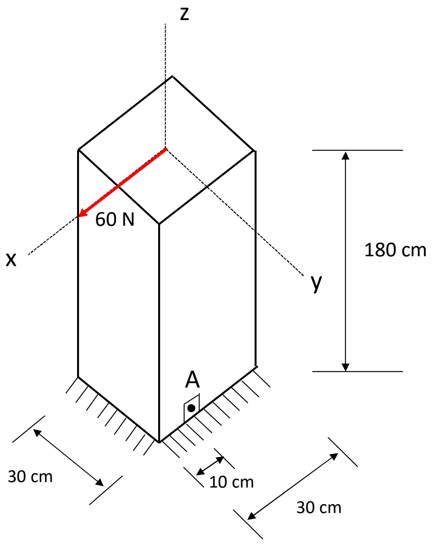
$\sigma_2 = 0\ MPa$

Principal state without shear

$\tau = 0\ MPa$



Example 2

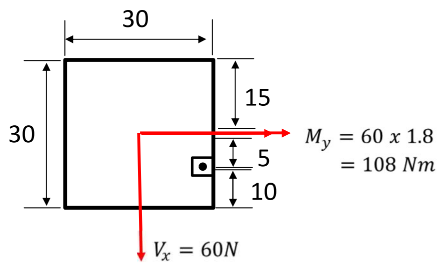


Determine

a) Max in-plane shear stress at A
b) The principal stress at A

The effect of the load should be transferred to the plane where A stands

(Cross-section at the bottom)

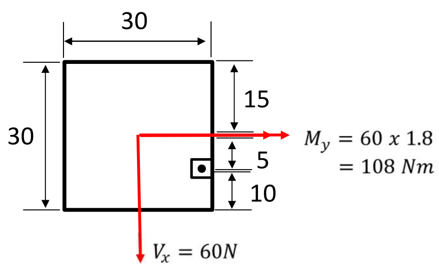


$$I_y = \frac{1}{12} 0.3 \; x \; (0.3)^3 = 6.75 \; x \; 10^{-6} \; m^4$$

$$\sigma_A = -\frac{108 \; x \; 0.05}{6.75 \; x \; 10^{-6}} = -0.8 \; x \; 10^6 \; Pa$$

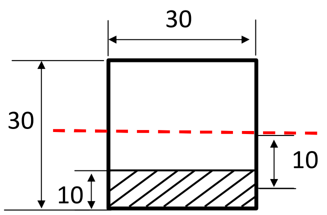
$$= -0.8 \; MPa$$

$$= -800 \; kPa$$



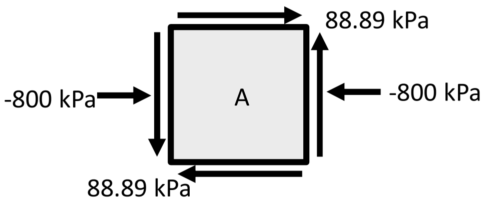
$$Q_A = (0.3 \; x \; 0.1) \; x \; 0.1$$

$$= 3 \; x \; 10^{-3} m^3$$



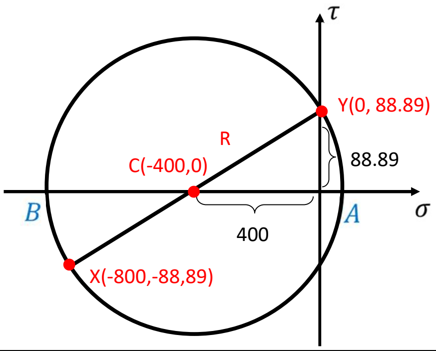
$$\tau_A = \frac{VQ}{It} = \frac{60 \; x \; (3 \; x \; 10^{-3})}{(6.75 \; x \; 10^{-6}) \; x \; 0.3}$$

$$\tau_A = 88.89 \; x \; 10^3 \; Pa = 88.89 \; kPa$$



$X(-800\,,-88.89)$

$Y(0,+88.89)$



$R = \sqrt{400^2 + 88.89^2}$

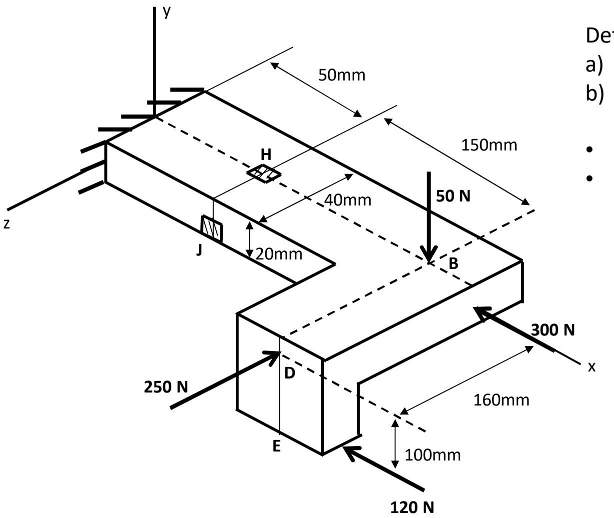
$R = 410\ kPa$

$\boxed{\tau_{max} = 410\ kPa}$

$\sigma_1 = -400 + 410 = 10\ kPa\ \ at\ \ A$

$\sigma_2 = -400 - 410 = -810\ kPa\ \ at\ \ B$

Example 3

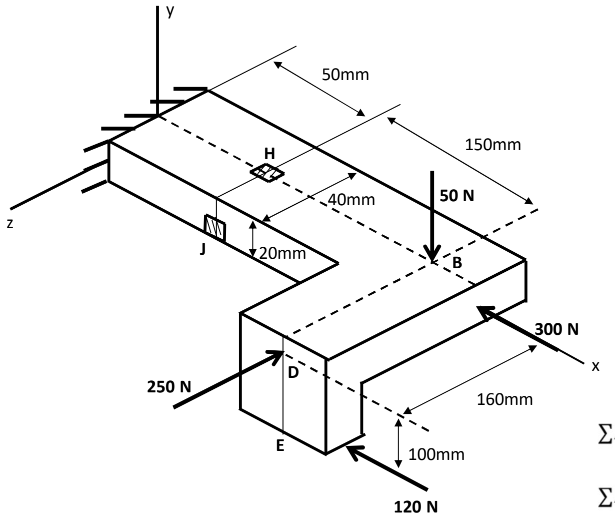


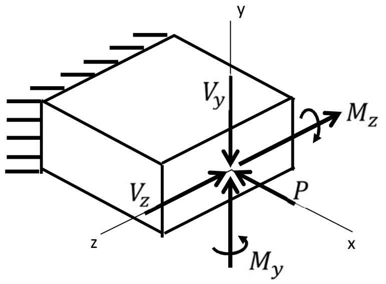
Determine normal and shearing stress at

a) point H
b) point J

- A combined loading example.
- Move the 4 loads with their moments to the plane that includes point H and J

The stresses are asked to be calculated on xyz axes. No rotation!





$\sum F_x \rightarrow \quad P = 120 + 300 \qquad \boldsymbol{P = 420\ N}$

$\sum F_y \rightarrow \quad \boldsymbol{V_y = 50\ N}$

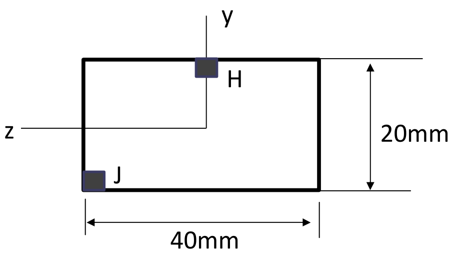
$\sum F_z \rightarrow \quad \boldsymbol{V_z = 250\ N}$

$\sum M_y \rightarrow \quad M_y = 250x0.15 - 120x0.16$

$\boldsymbol{M_y = +18.3\ Nm}$

$\sum M_z \rightarrow \quad M_z = 120x0.1 + 50x0.15$

$\boldsymbol{M_z = +19.5\ Nm}$



$A = 20x10^{-3}x40x10^{-3}$

$\boldsymbol{A = 8x10^{-4}\ m^2}$

$I_z = \frac{1}{12} 40x10^{-3}x(20x10^{-3})^3 \quad \boldsymbol{I_z = 2.67x10^{-8}\ m^4}$

$I_y = \frac{1}{12} 20x10^{-3}x(40x10^{-3})^3 \quad \boldsymbol{I_y = 1.07x10^{-7}\ m^4}$

a) point H

$$\sigma_H = \frac{-P}{A} + \frac{M_z x10x10^{-3}}{I_Z} = \frac{-420}{8x10^{-4}} + \frac{19.5x10x10^{-3}}{2.67x10^{-8}} = \boldsymbol{6.78\ MPa}$$

$$\tau_H = \frac{3}{2}\frac{V_z}{A} = \frac{3}{2}\frac{250}{8x10^{-4}} = \boldsymbol{0.47\ MPa}$$

V_y does not result in shear at H

b) point J

$$\sigma_J = \frac{-P}{A} + \frac{M_y x20x10^{-3}}{I_y} - \frac{M_z x10x10^{-3}}{I_z} = \frac{-420}{8x10^{-4}} + \frac{18.3x20x10^{-3}}{1.07x10^{-7}} - \frac{19.5x10x10^{-3}}{2.67x10^{-8}} = -4.41\ MPa$$

$\boldsymbol{\tau_J = 0}$ (J is at the corner)