ME 211 Statics and Strength of Materials

CHAPTER 3
Rigid bodies: Equivalent systems of forces

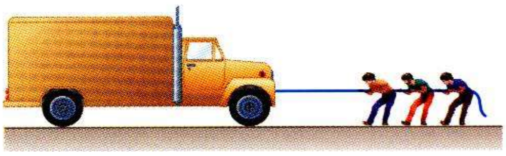
Contents

Introduction

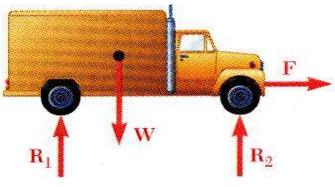
- Treatment of a body as a single **particle** is not always possible. In general, the **size of the body and the specific points of application of the forces must be considered**.
- Current chapter describes the effect of forces exerted on a rigid body **and how to replace a given system of forces with a simpler equivalent system**.
- First, we need to learn some new statics concepts, including:
  - **moment of a force about a point**
  - **moment of a force about an axis**
  - **moment due to a couple**
- **Any system of forces acting on a rigid body can be replaced by an equivalent system consisting of one force acting at a given point and one couple.**

External and Internal Forces

- Forces acting on rigid bodies are divided into two groups:
  - **External forces**
  - **Internal forces**



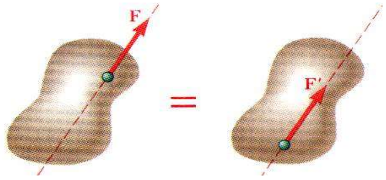
- External forces are shown in a **free body diagram**.



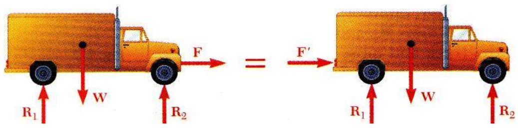
- Internal forces, such as the force **between each wheel and the axel it is mount on**, are never shown on a free body diagram.

Principle of Transmissibility: Equivalent Forces

- ***Principle of Transmissibility*** -
  Conditions of equilibrium or motion are not affected by *transmitting* a force along its line of action.
  NOTE: **F** and **F'** are equivalent forces.



- Moving the point of application of the force **F** to the rear bumper does not affect the motion or the other forces acting on the truck.

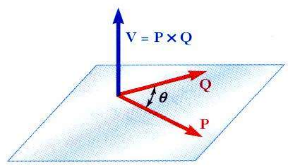


Vector Product of Two Vectors

- **Concept** of the <u>**moment of a force about a point**</u> requires the understanding of the ***vector product* or *cross product*.**

- Vector product of two vectors $\boldsymbol{P}$ and $\boldsymbol{Q}$ is defined as the vector $\boldsymbol{V}$ which satisfies the following conditions:
  1. Line of action of $\boldsymbol{V}$ is perpendicular to plane containing $\boldsymbol{P}$ and $\boldsymbol{Q}$.
  2. Magnitude of $\boldsymbol{V}$ is $V = PQ \sin\theta$
  3. Direction of $\boldsymbol{V}$ is obtained from the right-hand rule.

- Vector products:
  - are not commutative, $\boldsymbol{Q} \times \boldsymbol{P} = -(\boldsymbol{P} \times \boldsymbol{Q})$
  - are distributive, $\boldsymbol{P} \times (\boldsymbol{Q}_1 + \boldsymbol{Q}_2) = \boldsymbol{P} \times \boldsymbol{Q}_1 + \boldsymbol{P} \times \boldsymbol{Q}_2$
  - are not associative, $(\boldsymbol{P} \times \boldsymbol{Q}) \times \boldsymbol{S} \neq \boldsymbol{P} \times (\boldsymbol{Q} \times \boldsymbol{S})$



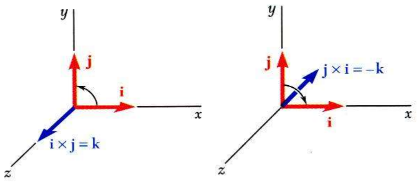
(a)

(b)

Vector Products: Rectangular Components

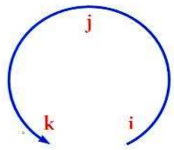
- Vector products of Cartesian unit vectors,

$$\vec{i}\times\vec{i}=0 \qquad \vec{j}\times\vec{i}=-\vec{k} \qquad \vec{k}\times\vec{i}=\vec{j}$$
$$\vec{i}\times\vec{j}=\vec{k} \qquad \vec{j}\times\vec{j}=0 \qquad \vec{k}\times\vec{j}=-\vec{i}$$
$$\vec{i}\times\vec{k}=-\vec{j} \qquad \vec{j}\times\vec{k}=\vec{i} \qquad \vec{k}\times\vec{k}=0$$



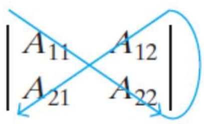
- Vector products in terms of rectangular coordinates (determinant of 3x3 matrix)

$$\vec{V}=\left(P_x\vec{i}+P_y\vec{j}+P_z\vec{k}\right)\times\left(Q_x\vec{i}+Q_y\vec{j}+Q_z\vec{k}\right)$$
$$=\left(P_yQ_z-P_zQ_y\right)\vec{i}+\left(P_zQ_x-P_xQ_z\right)\vec{j}$$
$$+\left(P_xQ_y-P_yQ_x\right)\vec{k}$$
$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ P_x & P_y & P_z\\ Q_x & Q_y & Q_z\end{vmatrix}$$

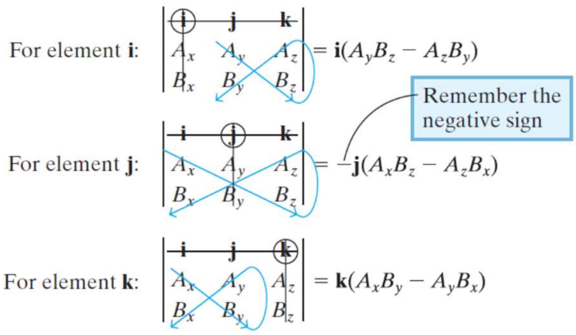


Vector Products: Rectangular Components

$$\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$



$$(A_{11}A_{22}-A_{12}A_{21})$$

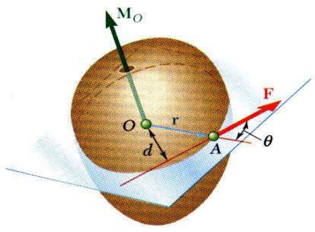


Moment of a Force About a Point

- A force vector is defined by its magnitude and direction. **Its effect on the rigid body also depends on its point of application**.
- The *moment* of ***F*** about *O* is defined as

$$\boldsymbol{M_O} = \boldsymbol{r} \times \boldsymbol{F}$$

- The moment vector $\boldsymbol{M_O}$ is perpendicular to the plane containing *O* and the force ***F***.
- Magnitude of $\boldsymbol{M_O}$, $M_O = rF \sin\theta = Fd$, **measures the tendency of the force to cause rotation of the body about an axis along M_O**. The sense of the moment may be determined by **the right-hand rule**.
- Any force ***F'*** that has the same magnitude and direction as ***F***, is *equivalent* if it also has the same line of action and therefore, produces the same moment.

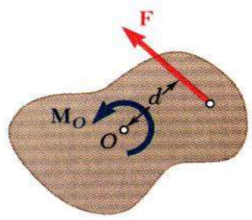


(a)

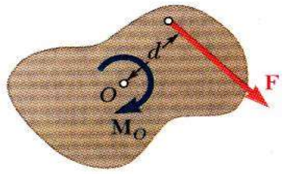
(b)

Moment of a Force About a Point

- *Two-dimensional structures* **have length and width** but **negligible depth** and are subjected to **forces** contained only **in the plane** of the structure.
- The plane of the structure contains the point *O* and the force ***F***. $\boldsymbol{M_O}$, the moment of the force about *O* is perpendicular to the plane.
- If the force tends to rotate the structure **counterclockwise**, the sense of the moment vector is out of the plane of the structure and the magnitude of the moment is **positive**.
- If the force tends to rotate the structure **clockwise**, the sense of the moment vector is into the plane of the structure and the magnitude of the moment is **negative**.

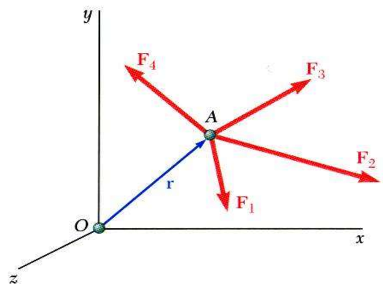


(a) $\mathbf{M}_O = +Fd$



(b) $\mathbf{M}_O = -Fd$

Varignon's Theorem



- **The moment about a give point *O* of the resultant of several concurrent forces is equal to the sum of the moments of the various moments about the same point *O*.**

$$\vec{r} \times \left(\vec{F}_1 + \vec{F}_2 + \cdots\right) = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2 + \cdots$$

- Varignon's Theorem makes it possible to replace the direct determination of the moment of a force ***F*** by the moments of two or more component forces of ***F***.

Rectangular Components of the Moment of a Force

The moment of ***F*** about *O*,

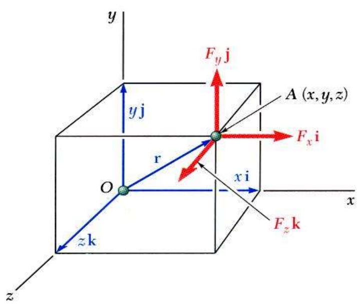
$$\vec{M}_O = \vec{r} \times \vec{F}, \qquad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$

$$\vec{M}_O = M_x\vec{i} + M_y\vec{j} + M_z\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \left(yF_z - zF_y\right)\vec{i} + \left(zF_x - xF_z\right)\vec{j} + \left(xF_y - yF_x\right)\vec{k}$$



The components of $\vec{M}_o$, M_x, M_y, and M_z, represent the moments about the x-, y- and z-axis, respectively.

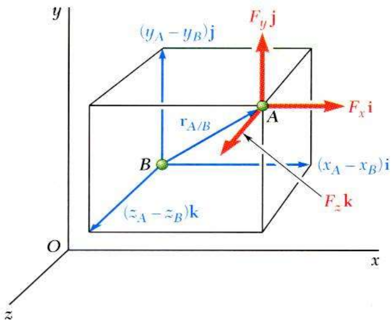
Rectangular Components of the Moment of a Force

The moment of ***F*** about *B*,

$$\vec{M}_B = \vec{r}_{A/B} \times \vec{F}$$

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B$$
$$= (x_A - x_B)\vec{i} + (y_A - y_B)\vec{j} + (z_A - z_B)\vec{k}$$
$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$

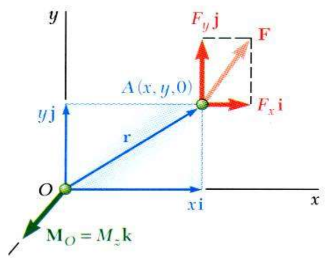
$$\vec{M}_B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ (x_A - x_B) & (y_A - y_B) & (z_A - z_B) \\ F_x & F_y & F_z \end{vmatrix}$$



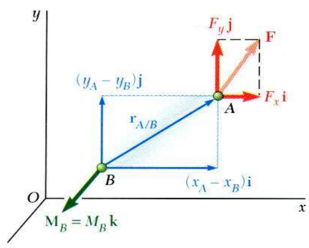
Rectangular Components of the Moment of a Force

For two-dimensional structures,

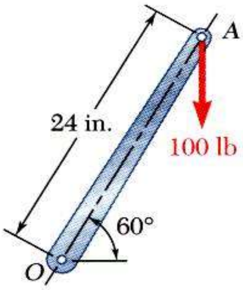
$$\vec{M}_O = (xF_y - yF_x)\vec{k}$$
$$M_O = M_Z$$
$$= xF_y - yF_x$$



$$\vec{M}_B = [(x_A - x_B)F_y - (y_A - y_B)F_x]\vec{k}$$
$$M_B = M_Z$$
$$= (x_A - x_B)F_y - (y_A - y_B)F_x$$



Sample Problem 3.1



A 100-lb vertical force is applied to the end of a lever which is attached to a shaft (not shown) at *O*.

Determine:

a) the moment about *O*,

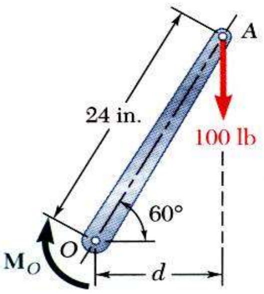
b) the horizontal force at *A* which creates the same moment,

c) the smallest force at A which produces the same moment,

d) the location for a 240-lb vertical force to produce the same moment,

e) whether any of the forces from b, c, and d is equivalent to the original force.

Sample Problem 3.1



a) Moment about *O* is equal to the product of the force and the perpendicular distance between the line of action of the force and *O*. Since the force tends to rotate the lever clockwise, the moment vector is into the plane of the paper which, by our sign convention, would be negative or counterclockwise.

$$|M_O| = Fd$$

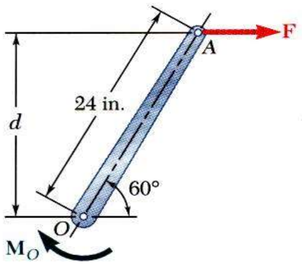
$$d = (24\text{ in.})\cos 60° = 12\text{ in.}$$

$$|M_O| = (100\text{ lb})(12\text{ in.})$$

$$M_O = -1200\text{ lb}\cdot\text{in, or}$$
$$= 1200\text{ lb}\cdot\text{in}$$

Sample Problem 3.1

b) Horizontal force at A that produces the same moment,



$$d = (24\ \text{in.})\sin 60° = 20.8\ \text{in.}$$

$$|M_O| = Fd$$

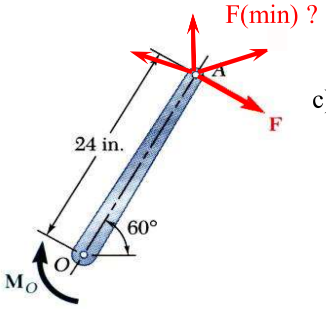
$$1200\ \text{lb}\cdot\text{in.} = F(20.8\ \text{in.})$$

$$F = \frac{1200\ \text{lb}\cdot\text{in.}}{20.8\ \text{in.}}$$

$$\boxed{F = 57.7\ \text{lb}}$$

Why must the direction of this F be to the right?

Sample Problem 3.1



What is the smallest force at A which produces the same moment? Think about it and discuss with a neighbor.

c) The smallest force at A to produce the same moment occurs when the perpendicular distance is a maximum or when F is perpendicular to OA.

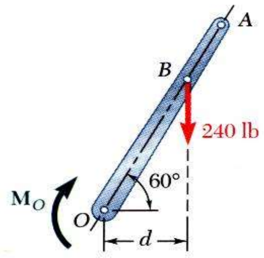
$$M_O = Fd$$

$$1200\ \text{lb}\cdot\text{in.} = F(24\ \text{in.})$$

$$F = \frac{1200\ \text{lb}\cdot\text{in.}}{24\ \text{in.}}$$

$$\boxed{F = 50\ \text{lb}}$$

Sample Problem 3.1



d) To determine the point of application of a 240 lb force to produce the same moment,

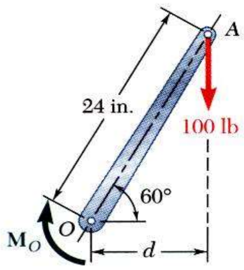
$$|M_O| = Fd$$

$$1200 \text{ lb} \cdot \text{in.} = (240 \text{ lb})d$$

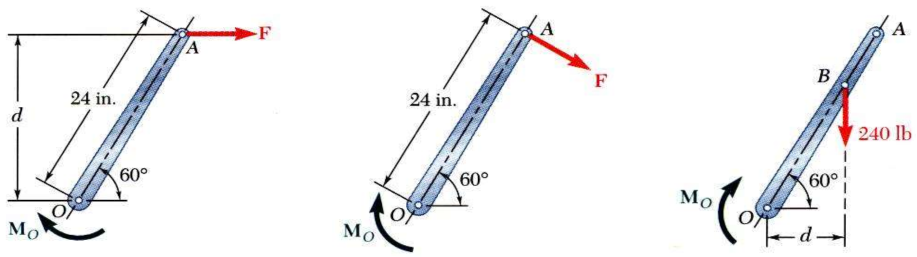
$$d = \frac{1200 \text{ lb} \cdot \text{in.}}{240 \text{ lb}} = 5 \text{ in.}$$

$$OB \cos 60° = 5 \text{ in.}$$

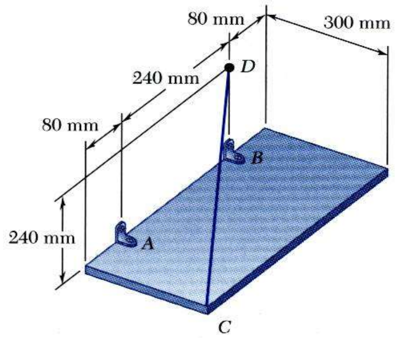
$$\boxed{OB = 10 \text{ in.}}$$



e) Although each of the forces in parts b), c), and d) produces the same moment as the 100 lb force, none are of the same magnitude and sense, or on the same line of action. None of the forces is equivalent to the 100 lb force.



Sample Problem 3.4



The rectangular plate is supported by the **brackets at *A* and *B*** and by a wire *CD*. Knowing that the **tension** in the wire is **200 N**, determine the **moment about *A*** of the force exerted by the wire at *C*.

<u>Solution</u>

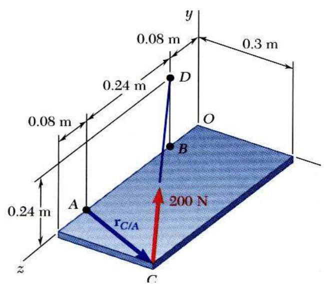
How do you find the moment? What is the equation for this moment? Discuss this with a neighbor and describe *in detail* each variable in your equation, including (a) how you would write it, (b) what its units are, and (c) whether or not there are any alternative variables that could take its place.

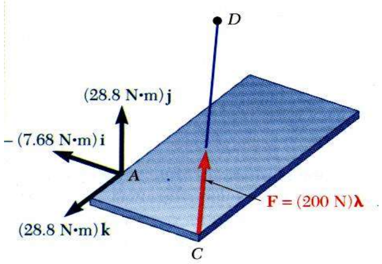
The moment M_A of the force F exerted by the wire is obtained by evaluating the vector product,

$$\vec{M}_A = \vec{r}_{C/A} \times \vec{F}$$

Sample Problem 3.4

<u>SOLUTION:</u>





$$\vec{M}_A = \vec{r}_{C/A} \times \vec{F}$$

$$\vec{r}_{C/A} = \vec{r}_C - \vec{r}_A = (0.3\text{ m})\vec{i} + (0.08\text{ m})\vec{j}$$

$$\vec{F} = F\vec{\lambda} = (200\text{ N})\frac{\vec{r}_{C/D}}{r_{C/D}}$$

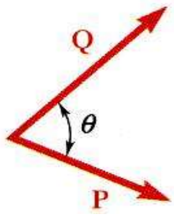
$$= (200\text{ N})\frac{-(0.3\text{ m})\vec{i} + (0.24\text{ m})\vec{j} - (0.32\text{ m})\vec{k}}{0.5\text{ m}}$$

$$= -(120\text{ N})\vec{i} + (96\text{ N})\vec{j} - (128\text{ N})\vec{k}$$

$$\vec{M}_A = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.3 & 0 & 0.08 \\ -120 & 96 & -128 \end{vmatrix}$$

$$\vec{M}_A = -(7.68\text{ N}\cdot\text{m})\vec{i} + (28.8\text{ N}\cdot\text{m})\vec{j} + (28.8\text{ N}\cdot\text{m})\vec{k}$$

Scalar Product of Two Vectors



- The ***scalar product*** **or** ***dot product*** between two vectors ***P*** and ***Q*** is defined as

  $\vec{P}\bullet\vec{Q} = PQ\cos\theta \quad (\text{scalar result})$
- Scalar products:
  - are commutative, $\vec{P}\bullet\vec{Q} = \vec{Q}\bullet\vec{P}$
  - are distributive, $\vec{P}\bullet\left(\vec{Q}_1 + \vec{Q}_2\right) = \vec{P}\bullet\vec{Q}_1 + \vec{P}\bullet\vec{Q}_2$
  - are not associative, $\left(\vec{P}\bullet\vec{Q}\right)\bullet\vec{S} = \text{undefined}$
- Scalar products with Cartesian unit components,

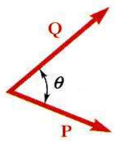
$$\vec{P}\bullet\vec{Q} = \left(P_x\vec{i} + P_y\vec{j} + P_z\vec{k}\right)\bullet\left(Q_x\vec{i} + Q_y\vec{j} + Q_z\vec{k}\right)$$

$$\vec{i}\bullet\vec{i} = 1 \quad \vec{j}\bullet\vec{j} = 1 \quad \vec{k}\bullet\vec{k} = 1 \quad \vec{i}\bullet\vec{j} = 0 \quad \vec{j}\bullet\vec{k} = 0 \quad \vec{k}\bullet\vec{i} = 0$$

$$\vec{P}\bullet\vec{Q} = P_xQ_x + P_yQ_y + P_zQ_z$$

$$\vec{P}\bullet\vec{P} = P_x^2 + P_y^2 + P_z^2 = P^2$$

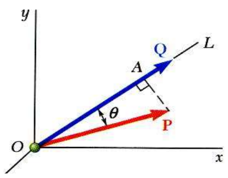
Scalar Product of Two Vectors: Applications



- **Angle between two vectors**:

$$\vec{P}\bullet\vec{Q} = PQ\cos\theta = P_xQ_x + P_yQ_y + P_zQ_z$$

$$\cos\theta = \frac{P_xQ_x + P_yQ_y + P_zQ_z}{PQ}$$

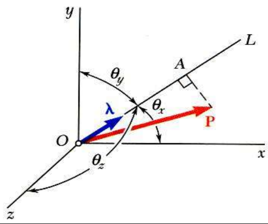


- **Projection of a vector on a given axis**:

$$P_{OL} = P\cos\theta = \text{projection of } P \text{ along } OL$$

$$\vec{P}\bullet\vec{Q} = PQ\cos\theta$$

$$\frac{\vec{P}\bullet\vec{Q}}{Q} = P\cos\theta = P_{OL}$$

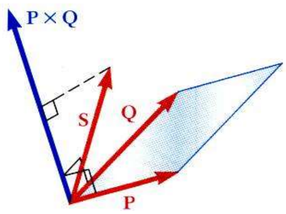


- For an axis defined by a unit vector:

$$P_{OL} = \vec{P}\bullet\vec{\lambda}$$

$$= P_x\cos\theta_x + P_y\cos\theta_y + P_z\cos\theta_z$$

Mixed Triple Product of Three Vectors



- Mixed triple product of three vectors,

  $\vec{S} \bullet \left(\vec{P} \times \vec{Q}\right) = \text{scalar result}$

- **The six mixed triple products formed from *S*, *P*, and *Q* have equal magnitudes but not the same sign,**

$$\vec{S} \bullet \left(\vec{P} \times \vec{Q}\right) = \vec{P} \bullet \left(\vec{Q} \times \vec{S}\right) = \vec{Q} \bullet \left(\vec{S} \times \vec{P}\right)$$
$$= -\vec{S} \bullet \left(\vec{Q} \times P\right) = -\vec{P} \bullet \left(\vec{S} \times \vec{Q}\right) = -\vec{Q} \bullet \left(\vec{P} \times \vec{S}\right)$$

- Evaluating the mixed triple product,

$$\vec{S} \bullet \left(\vec{P} \times \vec{Q}\right) = S_x\left(P_y Q_z - P_z Q_y\right) + S_y\left(P_z Q_x - P_x Q_z\right) + S_z\left(P_x Q_y - P_y Q_x\right)$$

$$= \begin{vmatrix} S_x & S_y & S_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}$$

Moment of a Force About a Given Axis

- Moment M_O of a force ***F*** applied at the point ***A*** about a point ***O***,

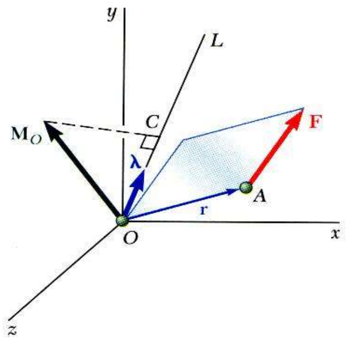
  $\vec{M}_O = \vec{r} \times \vec{F}$

- **Scalar moment M_{OL}** about an axis ***OL*** is the projection of the moment vector M_O onto the axis,

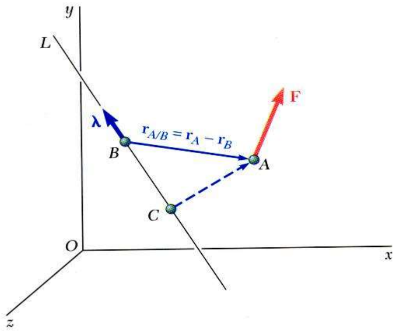
  $M_{OL} = \vec{\lambda} \bullet \vec{M}_O = \vec{\lambda} \bullet \left(\vec{r} \times \vec{F}\right)$

- Moments of ***F*** about the coordinate axes,

$$M_x = yF_z - zF_y$$
$$M_y = zF_x - xF_z$$
$$M_z = xF_y - yF_x$$



Moment of a Force About a Given Axis



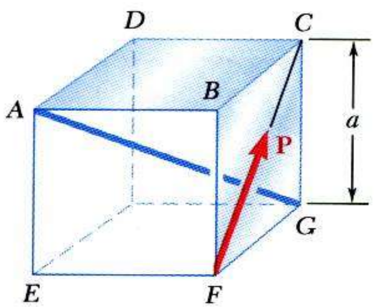
- Moment of a force about an arbitrary axis,

$$M_{BL} = \vec{\lambda} \bullet \vec{M}_B$$
$$= \vec{\lambda} \bullet \left(\vec{r}_{A/B} \times \vec{F}\right)$$

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B$$

- The result is independent of the point *B* along the given axis. That is, the same result can be gotten using $\vec{r}_{A/C}$, for example.

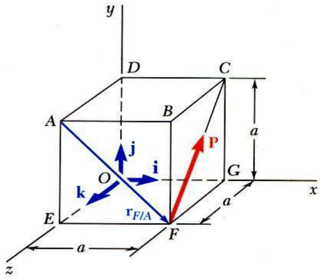
Sample Problem 3.5



A cube is acted on by a force ***P*** as shown. Determine the moment of ***P***

a) about *A*
b) about the edge *AB* and
c) about the diagonal *AG* of the cube

Sample Problem 3.5



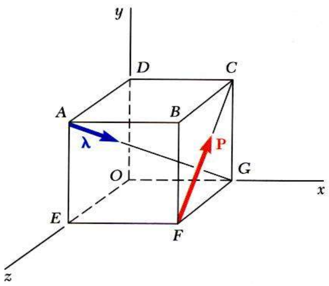
- Moment of ***P*** about *A*,

$$\vec{M}_A = \vec{r}_{F/A} \times \vec{P}$$
$$\vec{r}_{F/A} = a\vec{i} - a\vec{j} = a\left(\vec{i} - \vec{j}\right)$$
$$\vec{P} = P/\sqrt{2}\left(\vec{j} - \vec{k}\right)$$
$$\vec{M}_A = a\left(\vec{i} - \vec{j}\right) \times P/\sqrt{2}\left(\vec{j} - \vec{k}\right)$$
$$\boxed{\vec{M}_A = \left(aP\sqrt{2}\right)\left(\vec{i} + \vec{j} + \vec{k}\right)}$$

- Moment of ***P*** about *AB*,

$$M_{AB} = \vec{i} \bullet \vec{M}_A$$
$$= \vec{i} \bullet \left(aP\sqrt{2}\right)\left(\vec{i} + \vec{j} + \vec{k}\right)$$
$$\boxed{M_{AB} = aP\sqrt{2}}$$

Sample Problem 3.5



- Moment of ***P*** about the diagonal *AG*,

$$M_{AG} = \vec{\lambda} \bullet \vec{M}_A$$
$$\vec{\lambda} = \frac{\vec{r}_{G/A}}{r_{G/A}} = \frac{a\vec{i} - a\vec{j} - a\vec{k}}{a\sqrt{3}} = \frac{1}{\sqrt{3}}\left(\vec{i} - \vec{j} - \vec{k}\right)$$
$$\vec{M}_A = \frac{aP}{\sqrt{2}}\left(\vec{i} + \vec{j} + \vec{k}\right)$$
$$M_{AG} = \frac{1}{\sqrt{3}}\left(\vec{i} - \vec{j} - \vec{k}\right) \bullet \frac{aP}{\sqrt{2}}\left(\vec{i} + \vec{j} + \vec{k}\right)$$
$$= \frac{aP}{\sqrt{6}}\left(1 - 1 - 1\right)$$
$$\boxed{M_{AG} = -\frac{aP}{\sqrt{6}}}$$

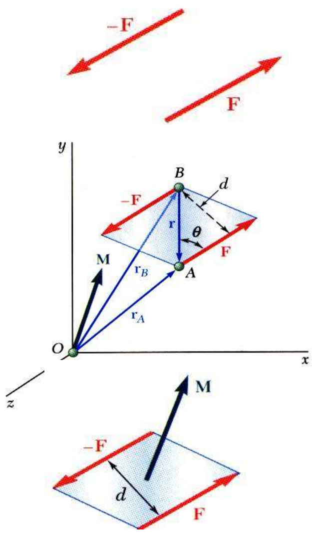
Moment of a Couple

- Two forces ***F*** and -***F*** having **the same magnitude, parallel lines of action, and opposite sense are said to form a *couple*.**

- Moment of the couple,

$$\begin{aligned} \vec{M} &= \vec{r}_A \times \vec{F} + \vec{r}_B \times \left(-\vec{F}\right) \\ &= \left(\vec{r}_A - \vec{r}_B\right) \times \vec{F} \\ &= \vec{r} \times \vec{F} \\ M &= rF \sin\theta = Fd \end{aligned}$$

- The moment vector of the couple is independent of the choice of the origin of the coordinate axes, i.e., it is a ***free vector* that can be applied at any point with the same effect**.



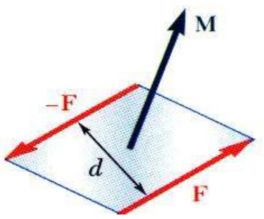
Moment of a Couple

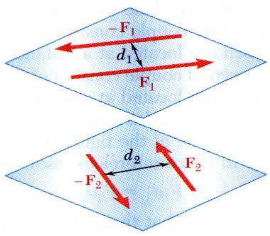
Two couples will have equal moments if

- $F_1 d_1 = F_2 d_2$

- the two couples lie in parallel planes, and

- the two couples have the same sense or the tendency to cause rotation in the same direction.





Addition of Couples

- Consider two intersecting planes P_1 and P_2 with each containing a couple

$$\vec{M}_1 = \vec{r} \times \vec{F}_1 \text{ in plane } P_1$$

$$\vec{M}_2 = \vec{r} \times \vec{F}_2 \text{ in plane } P_2$$

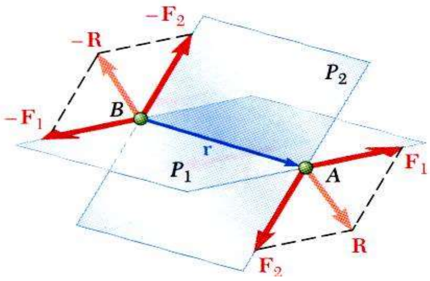
- Resultants of the vectors also form a couple

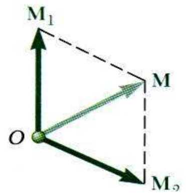
$$\vec{M} = \vec{r} \times \vec{R} = \vec{r} \times \left(\vec{F}_1 + \vec{F}_2\right)$$

- By Varignon's theorem

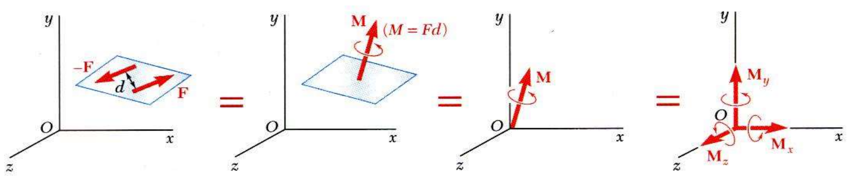
$$\vec{M} = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2$$
$$= \vec{M}_1 + \vec{M}_2$$

- Sum of two couples is also a couple that is equal to the vector sum of the two couples



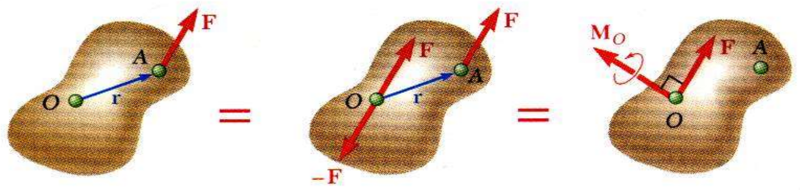


Couples Can Be Represented by Vectors



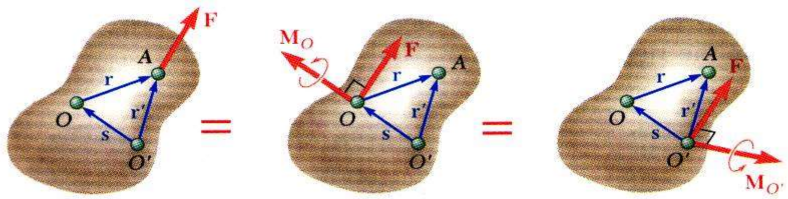
- **A couple can be represented by a vector with magnitude and direction equal to the moment of the couple**.
- *Couple vectors* obey the law of addition of vectors.
- **Couple vectors are free vectors**, i.e., there is no point of application – it simply acts on the body.
- **Couple vectors may be resolved into component vectors**.

Resolution of a Force Into a Force at *O* and a Couple



- **Force vector *F* can not be simply moved to *O* without modifying its action on the body.**
- Attaching equal and opposite force vectors at *O* produces no net effect on the body.
- The three forces may be replaced by an equivalent force vector and couple vector, i.e, a ***force-couple system***.

Resolution of a Force Into a Force at *O* and a Couple



- Moving ***F*** from *A* to a different point *O'* requires the addition of a different couple vector $M_{O'}$

$$\vec{M}_{O'} = \vec{r}\,' \times \vec{F}$$

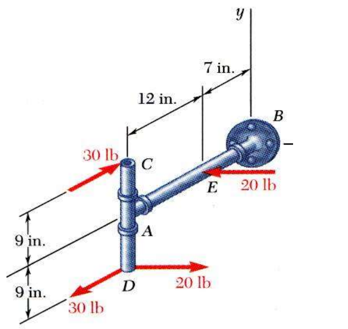
- The moments of ***F*** about O and *O'* are related,

$$\vec{M}_{O'} = \vec{r}\,'\times\vec{F} = (\vec{r} + \vec{s})\times\vec{F} = \vec{r}\times\vec{F} + \vec{s}\times\vec{F}$$

$$= \vec{M}_O + \vec{s}\times\vec{F}$$

- Moving the force-couple system from *O* to *O'* requires the addition of the moment of the force at *O* about *O'*.

Sample Problem 3.6

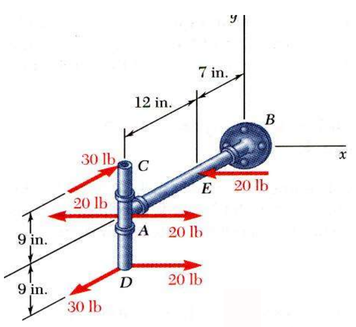


Determine the components of the single couple equivalent to the couples shown.

SOLUTION:

- Attach equal and opposite 20 lb forces in the $\pm x$ direction at A, thereby producing 3 couples for which the moment components are easily computed.
- Alternatively, compute the sum of the moments of the four forces about an arbitrary single point. The point D is a good choice as only two of the forces will produce non-zero moment contributions..

Sample Problem 3.6



- Attach equal and opposite 20 lb forces in the $\pm x$ direction at A
- The three couples may be represented by three couple vectors,

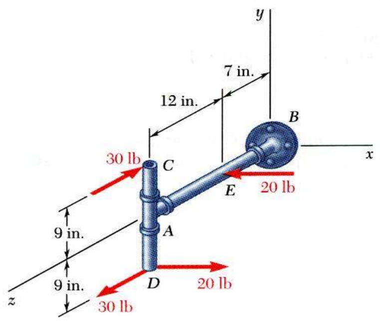
$$M_x = -(30\ \text{lb})(18\ \text{in.}) = -540\ \text{lb}\cdot\text{in.}$$

$$M_y = +(20\ \text{lb})(12\ \text{in.}) = +240\,\text{lb}\cdot\text{in.}$$

$$M_z = +(20\ \text{lb})(9\ \text{in.}) = +180\ \text{lb}\cdot\text{in.}$$

$$\vec{M} = -(540\ \text{lb}\cdot\text{in.})\vec{i} + (240\,\text{lb}\cdot\text{in.})\vec{j} + (180\ \text{lb}\cdot\text{in.})\vec{k}$$

Sample Problem 3.6

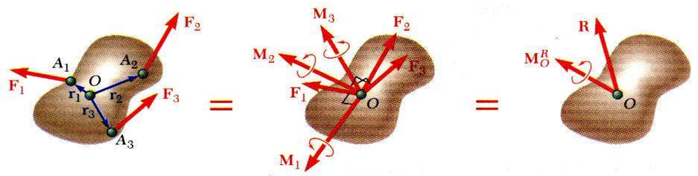


- Alternatively, compute the sum of the moments of the four forces about *D*.
- Only the forces at *C* and *E* contribute to the moment about *D*.

$$\vec{M} = \vec{M}_D = (18 \text{ in.})\vec{j} \times (-30 \text{ lb})\vec{k} + \left[(9 \text{ in.})\vec{j} - (12 \text{ in.})\vec{k}\right] \times (-20 \text{ lb})\vec{i}$$

$$\vec{M} = -(540 \text{ lb} \cdot \text{in.})\vec{i} + (240 \text{ lb} \cdot \text{in.})\vec{j} + (180 \text{ lb} \cdot \text{in.})\vec{k}$$

System of Forces: Reduction to a Force and Couple



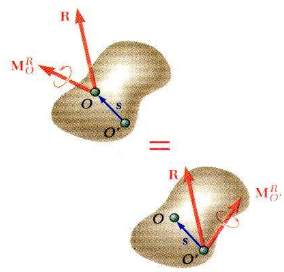
- **A system of forces may be replaced by a collection of force-couple systems acting at a given point *O***
- The force and couple vectors may be combined into a resultant force vector and a resultant couple vector,

$$\vec{R} = \sum \vec{F} \quad \vec{M}_O^R = \sum \left(\vec{r} \times \vec{F}\right)$$

- The force-couple system at *O* may be moved to *O'* with the addition of the moment of ***R*** about *O'* ,

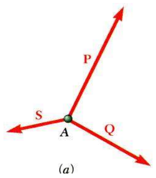
$$\vec{M}_{O'}^R = \vec{M}_O^R + \vec{s} \times \vec{R}$$

- Two systems of forces are equivalent if they can be reduced to the same force-couple system.

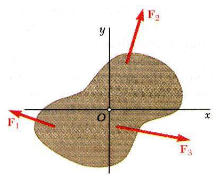


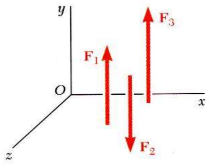
Further Reduction of a System of Forces

- If the **resultant force and couple at *O* are mutually perpendicular**, they can be replaced by **a single force** acting along a new line of action.
- The resultant force-couple system for a system of forces **will be mutually perpendicular if**:
  1) the forces are concurrent,
  2) the forces are coplanar, or
  3) the forces are parallel.

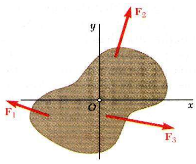


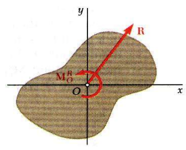
(a)

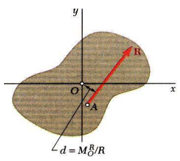




Further Reduction of a System of Forces

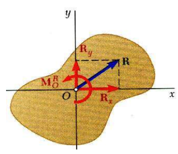


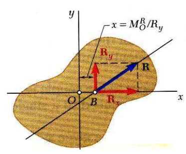


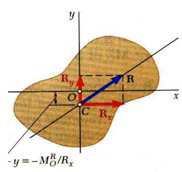


- System of coplanar forces is reduced to a force-couple system $\vec{R}$ and $\vec{M}_O^R$ that is mutually perpendicular.
- System can be reduced to a single force by moving the line of action of $\vec{R}$ until its moment about O becomes $\vec{M}_O^R$
- In terms of rectangular coordinates,

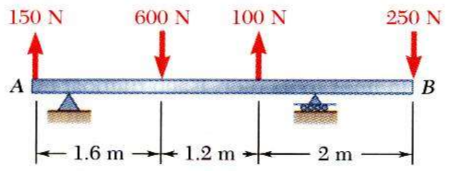
$$xR_y - yR_x = M_O^R$$







Sample Problem 3.8



For the beam, reduce the system of forces shown to (a) **an equivalent force-couple system at *A***, (b) an **equivalent force couple system at *B*.**

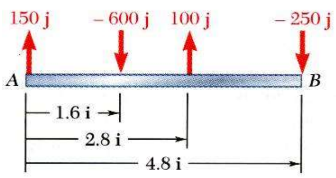
Note: **Since the support reactions are not included, the given system will not maintain the beam in equilibrium.**

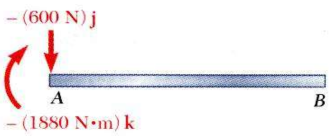
SOLUTION:

a) Compute the resultant force for the forces shown and the resultant couple for the moments of the forces about *A*.

b) Find an equivalent force-couple system at *B* based on the force-couple system at *A*.

Sample Problem 3.8





SOLUTION:

a) Compute the resultant force and the resultant couple at *A*.

$$\vec{R} = \sum \vec{F}$$

$$= (150\ \text{N})\vec{j} - (600\ \text{N})\vec{j} + (100\ \text{N})\vec{j} - (250\ \text{N})\vec{j}$$

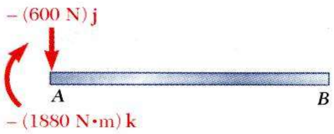
$$\boxed{\vec{R} = -(600\,\text{N})\vec{j}}$$

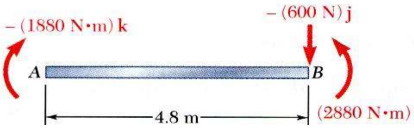
$$\vec{M}_A^R = \sum\left(\vec{r} \times \vec{F}\right)$$

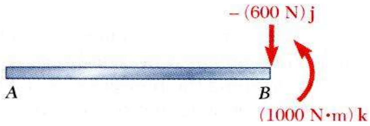
$$= \left(1.6\vec{i}\right)\times\left(-600\,\vec{j}\right) + \left(2.8\vec{i}\right)\times\left(100\,\vec{j}\right) + \left(4.8\vec{i}\right)\times\left(-250\,\vec{j}\right)$$

$$\boxed{\vec{M}_A^R = -(1880\ \text{N}\cdot\text{m})\vec{k}}$$

Sample Problem 3.8







b) Find an equivalent force-couple system at *B* based on the force-couple system at *A*.

The force is unchanged by the movement of the force-couple system from *A* to *B*.

$$\vec{R} = -(600\ \text{N})\vec{j}$$

The couple at *B* is equal to the moment about *B* of the force-couple system found at *A*.

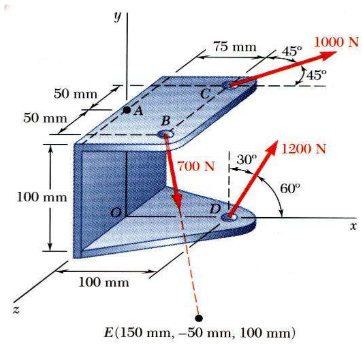
$$\vec{M}_B^R = \vec{M}_A^R + \vec{r}_{B/A} \times \vec{R}$$

$$= -(1880\ \text{N}\cdot\text{m})\vec{k} + (-4.8\ \text{m})\vec{i} \times (-600\ \text{N})\vec{j}$$

$$= -(1880\ \text{N}\cdot\text{m})\vec{k} + (2880\ \text{N}\cdot\text{m})\vec{k}$$

$$\vec{M}_B^R = +(1000\ \text{N}\cdot\text{m})\vec{k}$$

Sample Problem 3.10

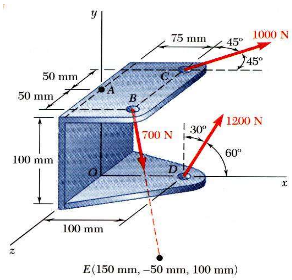


Three cables are attached to the bracket as shown. Replace the forces with an equivalent force-couple system at *A*.

SOLUTION:

- Determine the relative position vectors for the points of application of the cable forces with respect to *A*.
- Resolve the forces into rectangular components.
- Compute the equivalent force,
  $$\vec{R} = \sum \vec{F}$$
- Compute the equivalent couple,
  $$\vec{M}_A^R = \sum \left(\vec{r} \times \vec{F}\right)$$

Sample Problem 3.10



SOLUTION:

- Determine the relative position vectors with respect to *A*.

$$\vec{r}_{B/A} = 0.075\,\vec{i} + 0.050\,\vec{k}\ (\text{m})$$
$$\vec{r}_{C/A} = 0.075\,\vec{i} - 0.050\,\vec{k}\ (\text{m})$$
$$\vec{r}_{D/A} = 0.100\,\vec{i} - 0.100\,\vec{j}\ (\text{m})$$

- Resolve the forces into rectangular components.

$$\vec{F}_B = (700\ \text{N})\vec{\lambda}$$
$$\vec{\lambda} = \frac{\vec{r}_{E/B}}{r_{E/B}} = \frac{75\vec{i} - 150\vec{j} + 50\vec{k}}{175}$$
$$= 0.429\vec{i} - 0.857\vec{j} + 0.289\vec{k}$$
$$\vec{F}_B = 300\vec{i} - 600\vec{j} + 200\vec{k}\ (\text{N})$$

$$\vec{F}_C = (1000\ \text{N})(\cos 45\vec{i} - \cos 45\vec{j})$$
$$= 707\vec{i} - 707\vec{j}\ (\text{N})$$

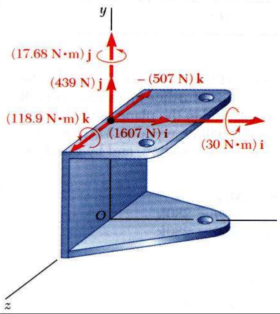
$$\vec{F}_D = (1200\ \text{N})(\cos 60\vec{i} + \cos 30\vec{j})$$
$$= 600\vec{i} + 1039\vec{j}\ (\text{N})$$

Sample Problem 3.10

- Compute the equivalent force,

$$\vec{R} = \sum \vec{F}$$
$$= (300 + 707 + 600)\vec{i} + (-600 + 1039)\vec{j} + (200 - 707)\vec{k}$$

$$\boxed{\vec{R} = 1607\,\vec{i} + 439\,\vec{j} - 507\,\vec{k}\ (\text{N})}$$



- Compute the equivalent couple,

$$\vec{M}_A^R = \sum(\vec{r} \times \vec{F})$$

$$\vec{r}_{B/A} \times \vec{F}_B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.075 & 0 & 0.050 \\ 300 & -600 & 200 \end{vmatrix} = 30\vec{i} - 45\vec{k}$$

$$\vec{r}_{C/A} \times \vec{F}_c = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.075 & 0 & -0.050 \\ 707 & 0 & -707 \end{vmatrix} = 17.68\vec{j}$$

$$\vec{r}_{D/A} \times \vec{F}_D = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.100 & -0.100 & 0 \\ 600 & 1039 & 0 \end{vmatrix} = 163.9\vec{k}$$

$$\boxed{\vec{M}_A^R = 30\,\vec{i} + 17.68\,\vec{j} + 118.9\,\vec{k}}$$