ME211 Statics and
Strength of Materials

CHAPTER 5
Distributed Forces: Centroids and Centers of Gravity
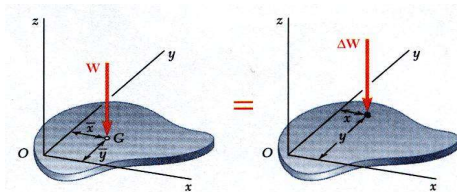
Contents

Application

There are many examples in engineering analysis of distributed loads. It is convenient in some cases to represent such loads as a concentrated force located at the *centroid* of the distributed load.

Introduction

- The **earth exerts a gravitational force on each of the particles forming a body** – consider how your weight is distributed throughout your body. These forces **can be replaced by a single equivalent force equal to the weight of the body and applied at the** *center of gravity* **for the body.**

- The *centroid of an area* is analogous to the **center of gravity of a body**; it is the "center of area." **The concept of the** *first moment of an area* **is used to locate the centroid.**

- Determination of the **area of a** *surface of revolution* and the **volume of a** *body of revolution* are accomplished with the *Theorems of Pappus-Guldinus*.

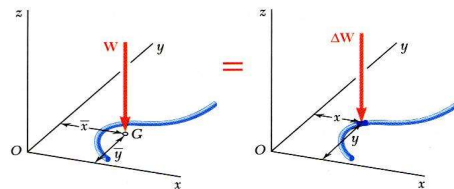## Center of Gravity of a 2D Body

- Center of gravity of a plate

- Center of gravity of a wire

$$\sum M_y \quad \overline{x}W = \sum x\Delta W$$
$$= \int x\, dW$$
$$\sum M_y \quad \overline{y}W = \sum y\Delta W$$
$$= \int y\, dW$$

$$\Sigma F_z: \quad W = \Delta W_1 + \Delta W_2 + \cdots + \Delta W_n$$

Centroids and First Moments of Areas and Lines

- Centroid of an area
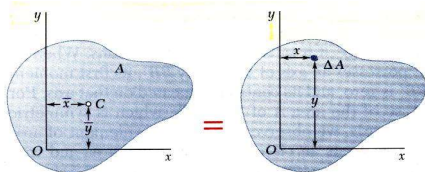
- Centroid of a line

$$\overline{x}W = \int x\, dW$$
$$\overline{x}(\gamma At) = \int x(\gamma t)dA$$
$$\overline{x}A = \int x\, dA = Q_y$$
$$\quad = \text{first moment wit h respect to } y$$
$$\overline{y}A = \int y\, dA = Q_x$$
$$\quad = \text{first moment wit h respect to } x$$
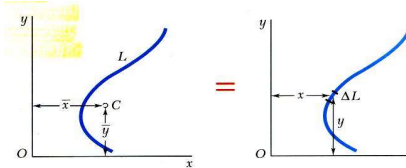
$$\overline{x}W = \int x\, dW$$
$$\overline{x}(\gamma La) = \int x(\gamma a)dL$$
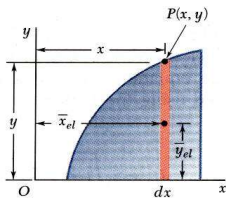$$\overline{x}L = \int x\, dL$$
$$\overline{y}L = \int y\, dL$$

$\gamma \rightarrow$ Weight per unit volume

Determination of Centroids by Integration

$$\bar{x}A = \int x\,dA = \iint x\,dx\,dy = \int \bar{x}_{el}\,dA$$

$$\bar{y}A = \int y\,dA = \iint y\,dx\,dy = \int \bar{y}_{el}\,dA$$

- **Double integration to find the first moment may be avoided by defining _dA_ as a thin rectangle or strip.**
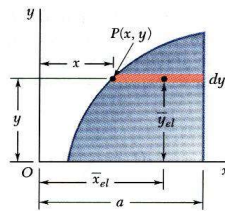

$$\bar{x}A = \int \bar{x}_{el}\,dA$$
$$= \int x\,(y\,dx)$$
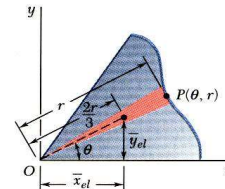$$\bar{y}A = \int \bar{y}_{el}\,dA$$
$$= \int \frac{y}{2}\,(y\,dx)$$

$$\bar{x}A = \int \bar{x}_{el}\,dA$$
$$= \int \frac{a+x}{2}\big[(a-x)\,dx\big]$$
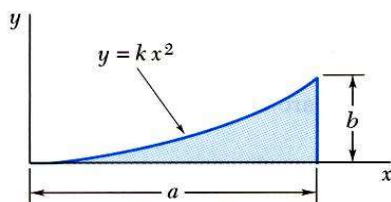$$\bar{y}A = \int \bar{y}_{el}\,dA$$
$$= \int y\,[(a-x)\,dx]$$

$$\bar{x}A = \int \bar{x}_{el}\,dA$$
$$= \int \frac{2r}{3}\cos\theta\left(\frac{1}{2}r^2\,d\theta\right)$$
$$\bar{y}A = \int \bar{y}_{el}\,dA$$
$$= \int \frac{2r}{3}\sin\theta\left(\frac{1}{2}r^2\,d\theta\right)$$
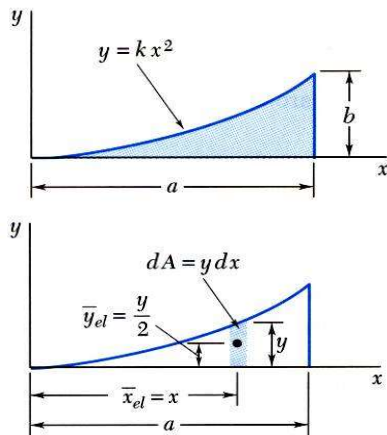

Sample Problem

Determine by direct integration the location of the centroid of a parabolic spandrel.

SOLUTION:

- Determine the constant k.

- Evaluate the total area.

- Using either vertical or horizontal strips, perform a single integration to find the first moments.

- Evaluate the centroid coordinates.

Sample Problem

SOLUTION:

- Determine the constant k.

$$y = k\,x^2$$

$$b = k\,a^2 \implies k = \frac{b}{a^2}$$

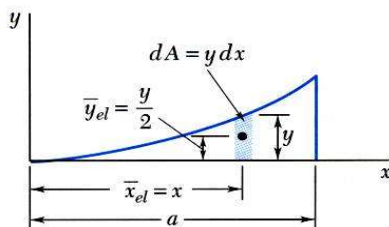$$y = \frac{b}{a^2}x^2 \quad or \quad x = \frac{a}{b^{1/2}}\,y^{1/2}$$

- Evaluate the total area.

$$A = \int dA$$

$$= \int y\,dx = \int_0^a \frac{b}{a^2}x^2\,dx = \left[\frac{b}{a^2}\frac{x^3}{3}\right]_0^a$$

$$= \frac{ab}{3}$$

Sample Problem

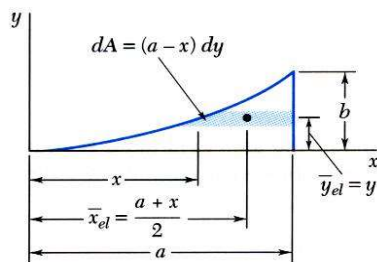- Using vertical strips, perform a single integration to find the first moments.

$$Q_y = \int \bar{x}_{el}\,dA = \int xy\,dx = \int_0^a x\left(\frac{b}{a^2}x^2\right)dx$$

$$= \left[\frac{b}{a^2}\frac{x^4}{4}\right]_0^a = \frac{a^2 b}{4}$$

$$Q_x = \int \bar{y}_{el}\,dA = \int \frac{y}{2}\,y\,dx = \int_0^a \frac{1}{2}\left(\frac{b}{a^2}x^2\right)^2 dx$$

$$= \left[\frac{b^2}{2a^4}\frac{x^5}{5}\right]_0^a = \frac{ab^2}{10}$$

Sample Problem

dA = (a − x) dy

$\overline{y}_{el} = y$

$\overline{x}_{el} = \dfrac{a + x}{2}$

- Or, using horizontal strips, perform a single integration to find the first moments. Try calculating Q_y or Q_x by this method, and confirm that you get the same value as before.
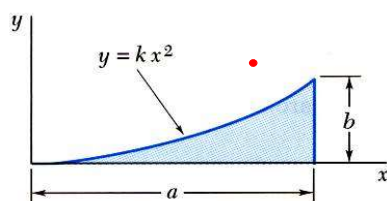
$$Q_y = \int \overline{x}_{el}\, dA = \int \frac{a + x}{2}(a - x)\, dy = \int_0^b \frac{a^2 - x^2}{2}\, dy$$

$$= \frac{1}{2}\int_0^b \left(a^2 - \frac{a^2}{b}y\right) dy = \frac{a^2 b}{4}$$

$$Q_x = \int \overline{y}_{el}\, dA = \int y(a - x)\, dy = \int y\left(a - \frac{a}{b^{1/2}}y^{1/2}\right) dy$$

$$= \int_0^b \left(ay - \frac{a}{b^{1/2}}y^{3/2}\right) dy = \frac{ab^2}{10}$$

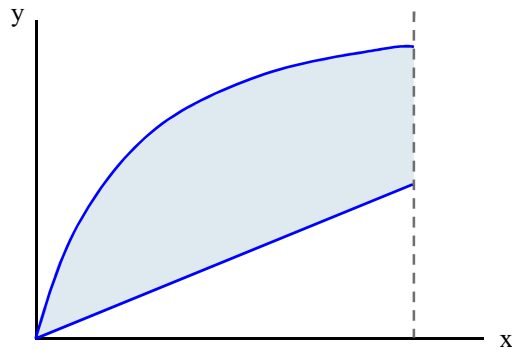## Sample Problem

$y = k\,x^2$

- Evaluate the centroid coordinates.

$$\overline{x}A = Q_y$$

$$\overline{x}\,\frac{ab}{3} = \frac{a^2 b}{4} \qquad\qquad \boxed{\overline{x} = \frac{3}{4}a}$$

$$\overline{y}A = Q_x$$

$$\overline{y}\,\frac{ab}{3} = \frac{ab^2}{10} \qquad\qquad \boxed{\overline{y} = \frac{3}{10}b}$$

Usually, the choice between using a vertical or horizontal strip is equally good, but in some cases, one choice is much better than the other. For example, for the area shown below, is a vertical or horizontal strip a better choice, and why?
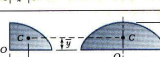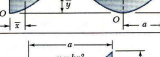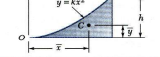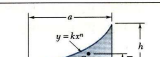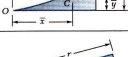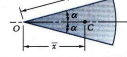

First Moments of Areas and Lines
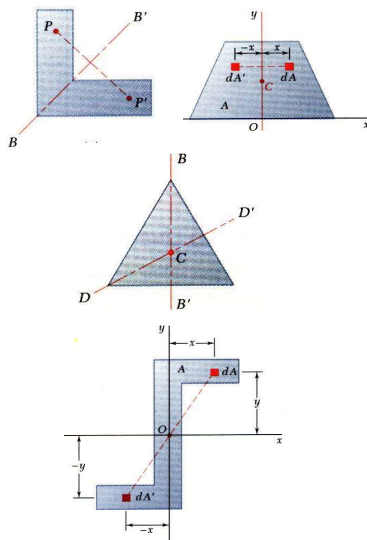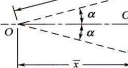
- An area is **symmetric with respect to an axis *BB'*** if for every point *P* there exists a point *P'* such that *PP'* is perpendicular to *BB'* and is divided into two equal parts by *BB'*.

- **The first moment of an area with respect to a line of symmetry is zero.**

- **If an area possesses a line of symmetry, its centroid lies on that axis**

- **If an area possesses two lines of symmetry, its centroid lies at their intersection.**

- **An area is symmetric with respect to a center *O* if for every element *dA* at (*x,y*) there exists an area *dA'* of equal area at (*-x,-y*).**

- <u>**The centroid of the area coincides with the center of symmetry.**</u>

Centroids of Common Shapes of Areas

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel	$y = kx^2$	$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel	$y = kx^n$	$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r\sin\alpha}{3\alpha}$	0	αr^2

Centroids of Common Shapes of Lines

Shape		$\bar{x}$	$\bar{y}$	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r\sin\alpha}{\alpha}$	0	$2\alpha r$

Composite Plates and Areas
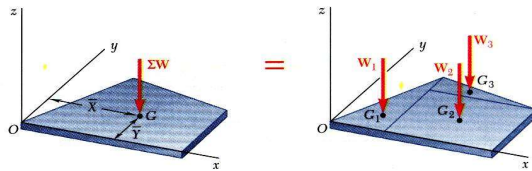
- Composite plates

$$\overline{X}\sum W = \sum \overline{x}\,W$$
$$\overline{Y}\sum W = \sum \overline{y}\,W$$
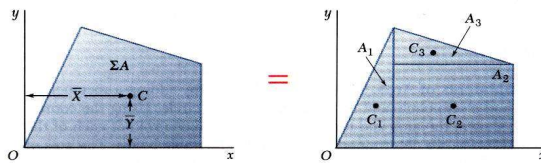
- Composite area

$$\overline{X}\sum A = \sum \overline{x}\,A$$
$$\overline{Y}\sum A = \sum \overline{y}\,A$$

Sample Problem
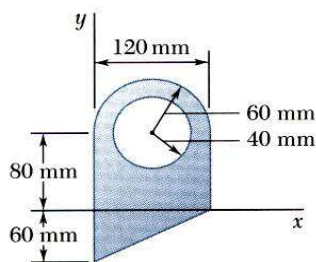
For the plane area shown, determine the first moments with respect to the *x* and *y* axes and the location of the centroid.

SOLUTION:

- Divide the area into a triangle, rectangle, and semicircle with a circular cutout.

- Calculate the first moments of each area with respect to the axes.

- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.

- Compute the coordinates of the area centroid by dividing the first moments by the total area.

Sample Problem
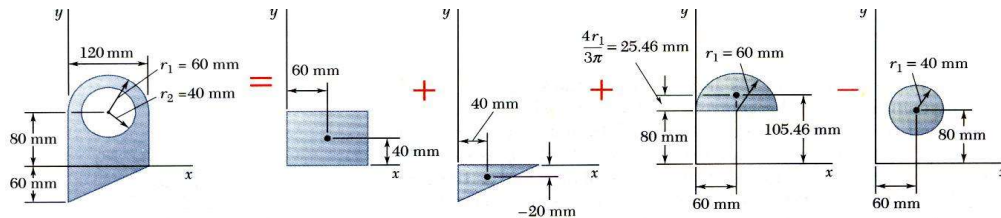
| Component	A, mm²	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm³	$\bar{y}A$, mm³
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	-20	$+144 \times 10^3$	-72×10^3
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	-301.6×10^3	-402.2×10^3
	$\Sigma A = 13.828 \times 10^3$			$\Sigma\bar{x}A = +757.7 \times 10^3$	$\Sigma\bar{y}A = +506.2 \times 10^3$

- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.

$$Q_x = +506.2 \times 10^3 \text{ mm}^3$$
$$Q_y = +757.7 \times 10^3 \text{ mm}^3$$

Sample Problem

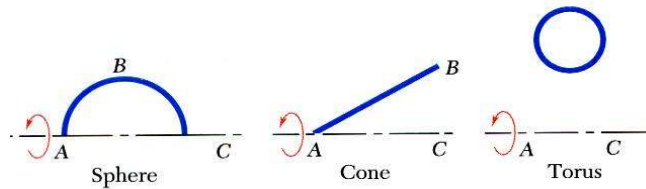- Compute the coordinates of the area centroid by dividing the first moments by the total area.
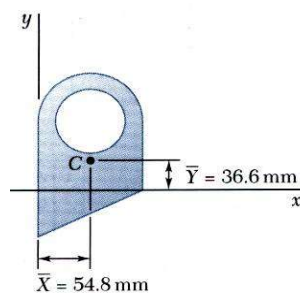
$$\bar{X} = \frac{\sum \bar{x}A}{\sum A} = \frac{+757.7 \times 10^3 \text{ mm}^3}{13.828 \times 10^3 \text{ mm}^2}$$

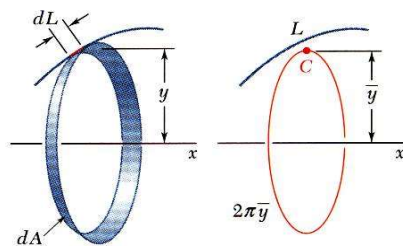$$\boxed{\bar{X} = 54.8 \text{ mm}}$$

$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{+506.2 \times 10^3 \text{ mm}^3}{13.828 \times 10^3 \text{ mm}^2}$$

$$\boxed{\bar{Y} = 36.6 \text{ mm}}$$

Theorems of Pappus-Guldinus

- Surface of revolution is generated by rotating a plane curve about a fixed axis.

- Area of a surface of revolution is equal to the **length of the generating curve** times the **distance traveled by the centroid through the rotation**.

$$A = 2\pi \bar{y} L$$

Theorems of Pappus-Guldinus
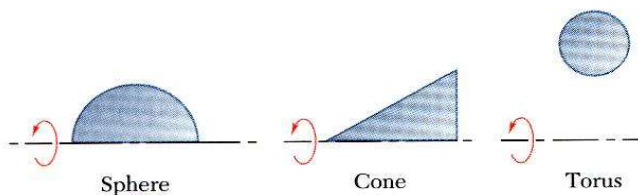
- Body of revolution is generated by rotating a plane area about a fixed axis.

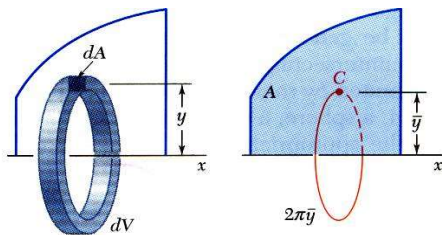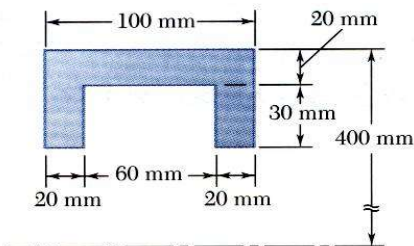- Volume of a body of revolution is equal to the generating area times the **distance traveled by the centroid through the rotation.**

$$V = 2\pi \bar{y} A$$

Sample Problem

100 mm · 20 mm · 30 mm · 400 mm · 60 mm · 20 mm · 20 mm

The outside diameter of a pulley is 0.8 m, and the cross section of its rim is as shown. Knowing that the pulley is made of steel and that the density of steel is $\rho = 7.85 \times 10^3 \ \text{kg}/\text{m}^3$ determine the mass and weight of the rim.

SOLUTION:

- Apply the theorem of Pappus-Guldinus to evaluate the volumes of revolution of the pulley, which we will form as a large rectangle with an inner rectangular cutout.

- Multiply by density and acceleration to get the mass and weight.

SOLUTION:

- Apply the theorem of **Pappus-Guldinus to evaluate the volumes or revolution for the rectangular rim section and the inner cutout section.**

- Multiply by density and acceleration to get the mass and weight.

100 mm · 60 mm · 50 mm · C_I · 30 mm · II · C_{II} · 375 mm · 365 mm
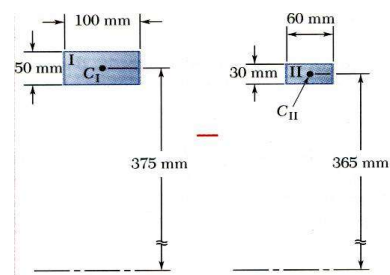
	Area, mm²	$\bar{y}$, mm	Distance Traveled by C, mm	Volume, mm³
I	+5000	375	$2\pi(375) = 2356$	$(5000)(2356) = 11.78 \times 10^6$
II	−1800	365	$2\pi(365) = 2293$	$(-1800)(2293) = -4.13 \times 10^6$
				Volume of rim $= 7.65 \times 10^6$

$$m = \rho V = \left(7.85 \times 10^3 \ \text{kg}/\text{m}^3\right)\left(7.65 \times 10^6 \text{mm}^3\right)\left(10^{-9} \text{m}^3/\text{mm}^3\right)$$

$$W = mg = \left(60.0 \ \text{kg}\right)\left(9.81 \ \text{m}/\text{s}^2\right)$$

$m = 60.0 \text{ kg}$

$W = 589 \text{ N}$

Distributed Loads on Beams
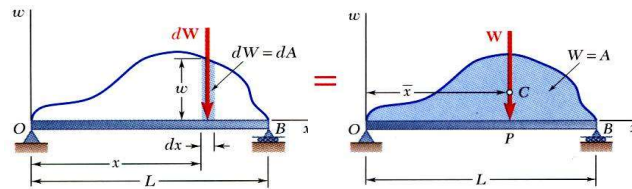
$$W = \int_0^L wdx = \int dA = A$$
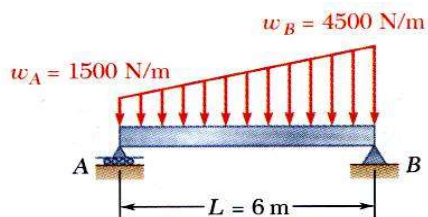
$$(OP)W = \int xdW$$

$$(OP)A = \int_0^L xdA = \bar{x}A$$

- **A distributed load is represented by plotting the load per unit length, *w* (N/m) . The total load is equal to the _area_ under the load curve.**

- A distributed load can be replace by a **concentrated load with a magnitude equal to the area under the load curve and a line of action passing through the area centroid**.


Sample Problem

A beam supports a distributed load as shown. Determine the equivalent concentrated load and the reactions at the supports.

SOLUTION:

- The magnitude of the concentrated load is equal to the total load or the area under the curve.

- The line of action of the concentrated load passes through the centroid of the area under the curve.

- Determine the support reactions by (a) drawing the free body diagram for the beam and (b) applying the conditions of equilibrium.
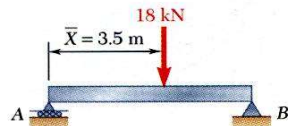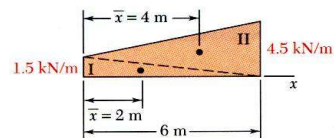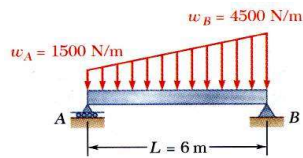
SOLUTION:

- The magnitude of the concentrated load is equal to the **total load or the area under the curve**.
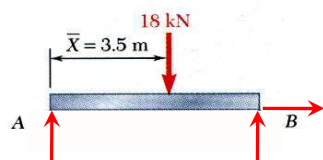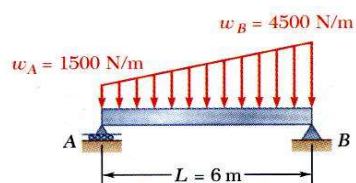
$$\boxed{F = 18.0 \text{ kN}}$$

- The line of action of the concentrated load passes through the centroid of the area under the curve.

$$\overline{X} = \frac{63 \text{ kN} \cdot \text{m}}{18 \text{ kN}} \qquad \boxed{\overline{X} = 3.5 \text{ m}}$$

Component	A, kN	$\bar{x}$, m	$\bar{x}A$, kN·m
Triangle I	4.5	2	9
Triangle II	13.5	4	54
	$\Sigma A = 18.0$		$\Sigma \bar{x}A = 63$

- Determine the support reactions by applying the equilibrium conditions. For example, successively sum the moments at the two supports:

$$\sum M_A = 0: \quad B_y(6 \text{ m}) - (18 \text{ kN})(3.5 \text{ m}) = 0$$
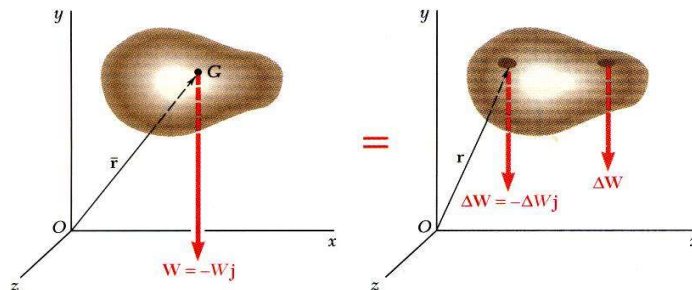
$$\boxed{B_y = 10.5 \text{ kN}}$$

$$\sum M_B = 0: \quad -A_y(6 \text{ m}) + (18 \text{ kN})(6 \text{ m} - 3.5 \text{ m}) = 0$$

$$\boxed{A_y = 7.5 \text{ kN}}$$

- And by summing forces in the x-direction:

$$\sum F_x = 0: \quad \boxed{B_x = 0}$$

Center of Gravity of a 3D Body: Centroid of a Volume

- Center of gravity G

$$-W\,\vec{j} = \sum\left(-\Delta W\,\vec{j}\right)$$

$$\vec{r}_G \times\left(-W\,\vec{j}\right) = \sum\left[\vec{r}\times\left(-\Delta W\,\vec{j}\right)\right]$$
$$\vec{r}_G W \times\left(-\vec{j}\right) = \left(\sum \vec{r}\,\Delta W\right)\times\left(-\vec{j}\right)$$

$$W = \int dW \qquad \vec{r}_G W = \int \vec{r}\,dW$$
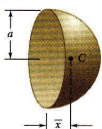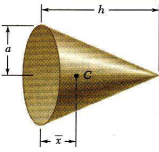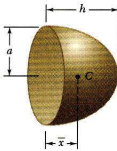
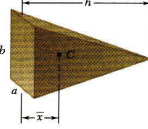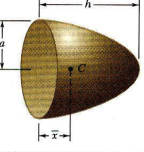- Results are independent of body orientation,

$$\bar{x}W = \int x\,dW \qquad \bar{y}W = \int y\,dW \qquad \bar{z}W = \int z\,dW$$

- For homogeneous bodies,

$$W = \gamma V \quad\text{and}\quad dW = \gamma\,dV$$

$$\bar{x}V = \int x\,dV \qquad \bar{y}V = \int y\,dV \qquad \bar{z}V = \int z\,dV$$

Centroids of Common 3D Shapes

Shape		$\bar{x}$	Volume				$\bar{x}$	Volume
Hemisphere		$\frac{3a}{8}$	$\frac{2}{3}\pi a^3$	Cone			$\frac{h}{4}$	$\frac{1}{3}\pi a^2 h$
Semiellipsoid of revolution		$\frac{3h}{8}$	$\frac{2}{3}\pi a^2 h$	Pyramid			$\frac{h}{4}$	$\frac{1}{3}abh$
Paraboloid of revolution		$\frac{h}{3}$	$\frac{1}{2}\pi a^2 h$					

Composite 3D Bodies
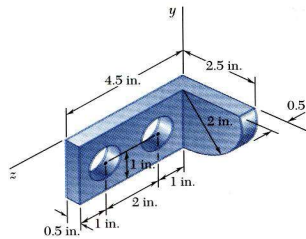
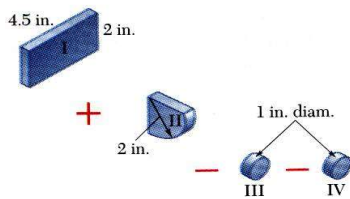- Moment of the total weight concentrated at the center of gravity G is equal to the sum of the moments of the weights of the component parts.
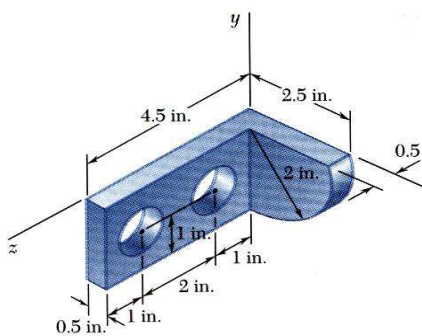
$$\overline{X}\sum W = \sum \overline{x}W \quad \overline{Y}\sum W = \sum \overline{y}W \quad \overline{Z}\sum W = \sum \overline{z}W$$

- For homogeneous bodies,

$$\overline{X}\sum V = \sum \overline{x}V \quad \overline{Y}\sum V = \sum \overline{y}V \quad \overline{Z}\sum V = \sum \overline{z}V$$
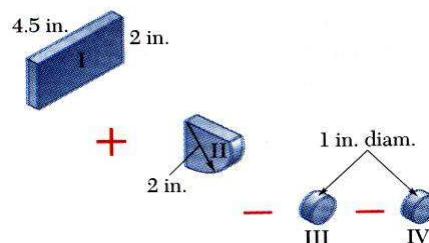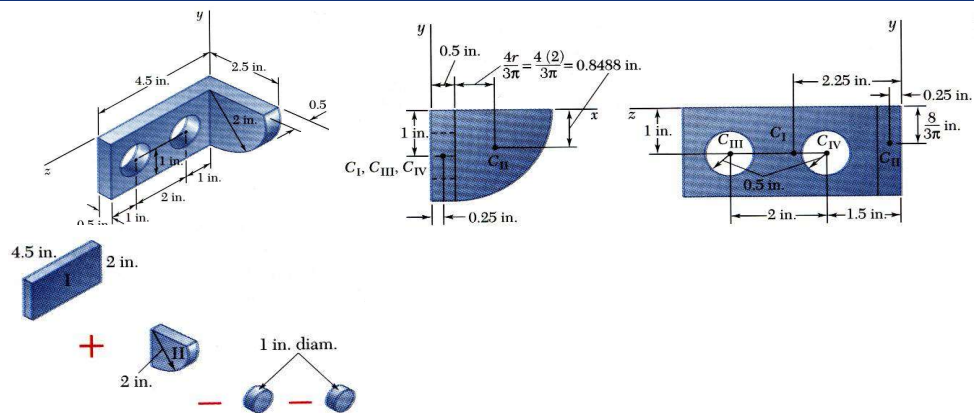
Sample Problem

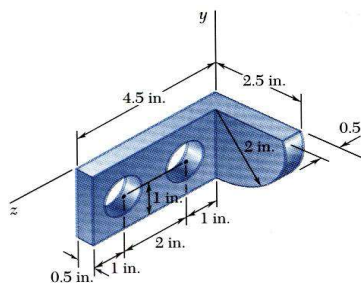Locate the center of gravity of the steel machine element. The diameter of each hole is 1 in.

SOLUTION:

- Form the machine element from a rectangular parallelepiped and a quarter cylinder and then subtracting two 1-in. diameter cylinders.

	V, in³	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{x}V$, in⁴	$\bar{y}V$, in⁴	$\bar{z}V$, in⁴
I	$(4.5)(2)(0.5) = 4.5$	0.25	−1	2.25	1.125	−4.5	10.125
II	$\frac{1}{4}\pi(2)^2(0.5) = 1.571$	1.3488	−0.8488	0.25	2.119	−1.333	0.393
III	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	−1	3.5	−0.098	0.393	−1.374
IV	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	−1	1.5	−0.098	0.393	−0.589
	$\Sigma V = 5.286$				$\Sigma\bar{x}V = 3.048$	$\Sigma\bar{y}V = -5.047$	$\Sigma\bar{z}V = 8.555$

	V, in³	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{x}V$, in⁴	$\bar{y}V$, in⁴	$\bar{z}V$, in⁴
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	$\Sigma V = 5.286$				$\Sigma\bar{x}V = 3.048$	$\Sigma\bar{y}V = -5.047$	$\Sigma\bar{z}V = 8.555$

$$\overline{X} = \sum \bar{x}V \Big/ \sum V = \left(3.08 \text{ in}^4\right)\Big/\left(5.286 \text{ in}^3\right)$$

$$\boxed{\overline{X} = 0.577 \text{ in.}}$$

$$\overline{Y} = \sum \bar{y}V \Big/ \sum V = \left(-5.047 \text{ in}^4\right)\Big/\left(5.286 \text{ in}^3\right)$$

$$\boxed{\overline{Y} = 0.577 \text{ in.}}$$

$$\overline{Z} = \sum \bar{z}V \Big/ \sum V = \left(1.618 \text{ in}^4\right)\Big/\left(5.286 \text{ in}^3\right)$$

$$\boxed{\overline{Z} = 0.577 \text{ in.}}$$