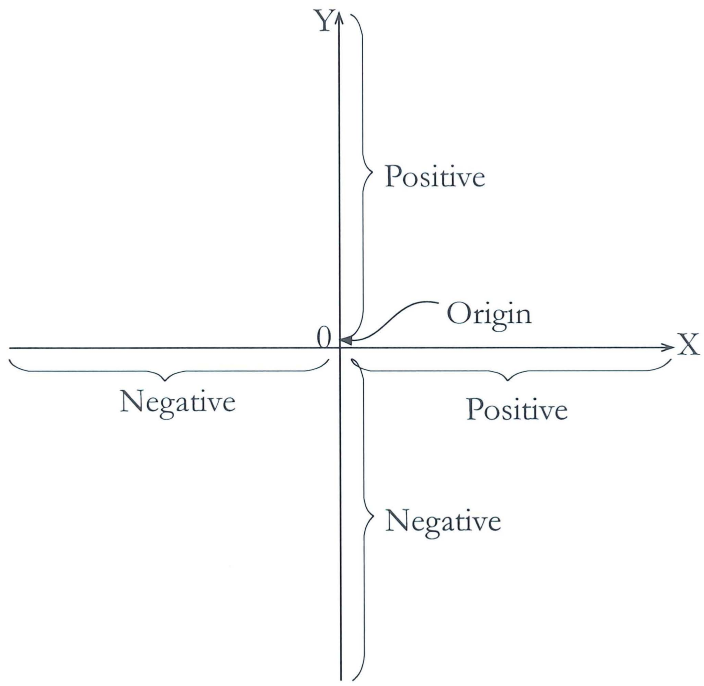
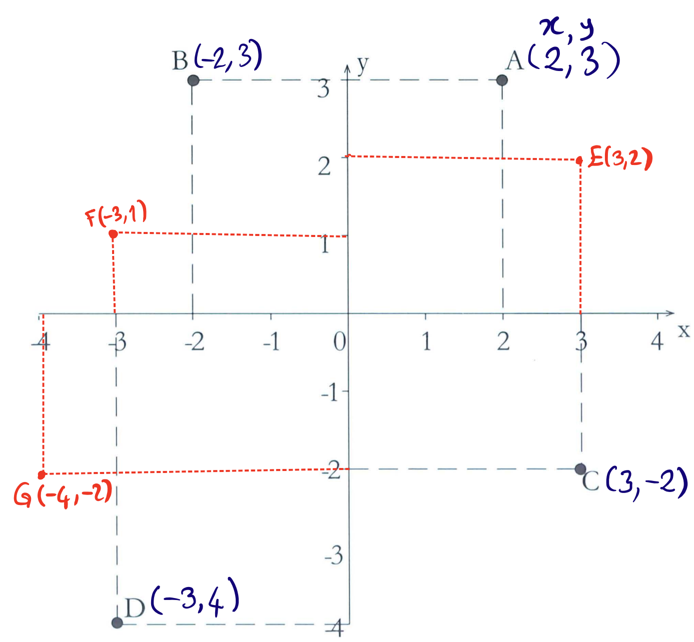
Graphs (Introduction)

Every graph shows a relationship between two sets of figures. To plot a graph, the first step is to draw the ***axes of reference***. The point where these two axes intersect is called the ***origin***. The ***horizontal axis*** is usually referred to as the ***x-axis*** and the ***vertical one*** as the ***y-axis***.



On the x-axis, the part to the right of the origin is ***positive*** and that to the left is ***negative***. On the y-axis, the part above the origin ***is positive*** and that below the origin ***is negative***.

Coordinates are used to mark individual points on a graph. In the following diagram the point A has been plotted so that x = 2 and y = 3, point B so that x = – 2 and y = 3, point C so that x = 3 and y = – 2 and point D so that x = – 3 and y = – 4



E(3,2)
F(-3,1)
G(-4,-2)

The values of x and y for the points are called ***rectangular coordinates***. So, the rectangular coordinates of A for example are 2 and 3.

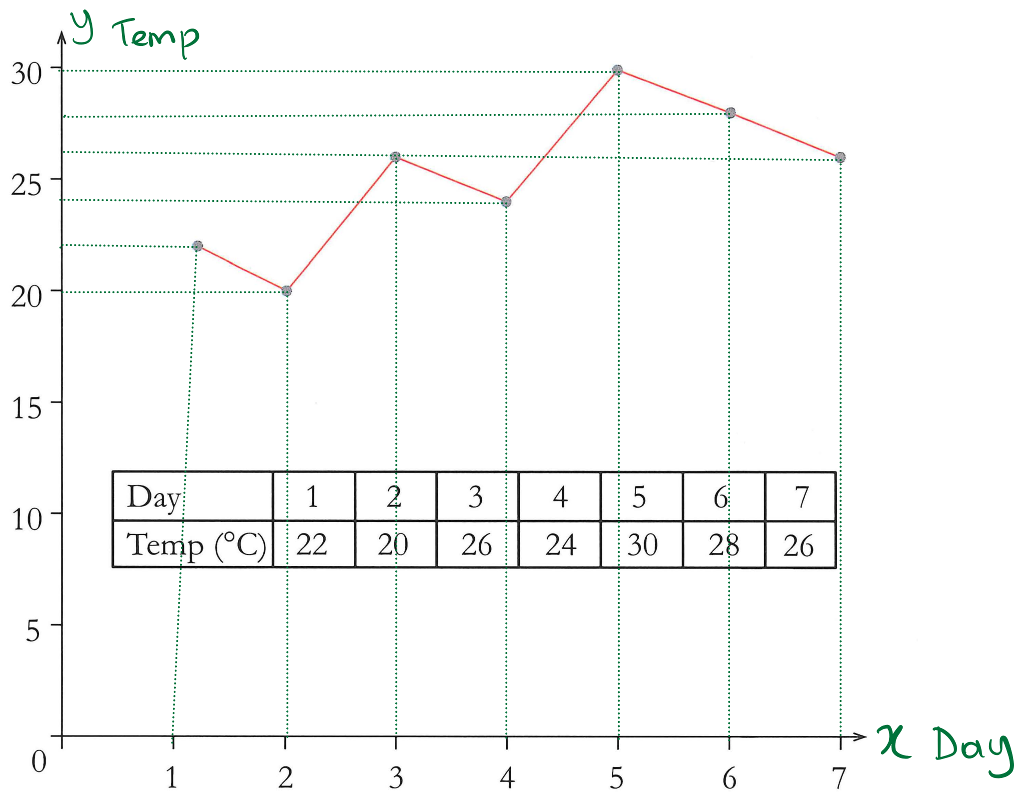
These coordinates are always shown in brackets separated by a comma, so the coordinates of A would be (2,3).

It is important to note that when ***coordinates*** are given the ***x value must always be given first***.

Example

The following table gives the midday temperature on seven successive days at a holiday resort.

Plot a line graph to show this information.



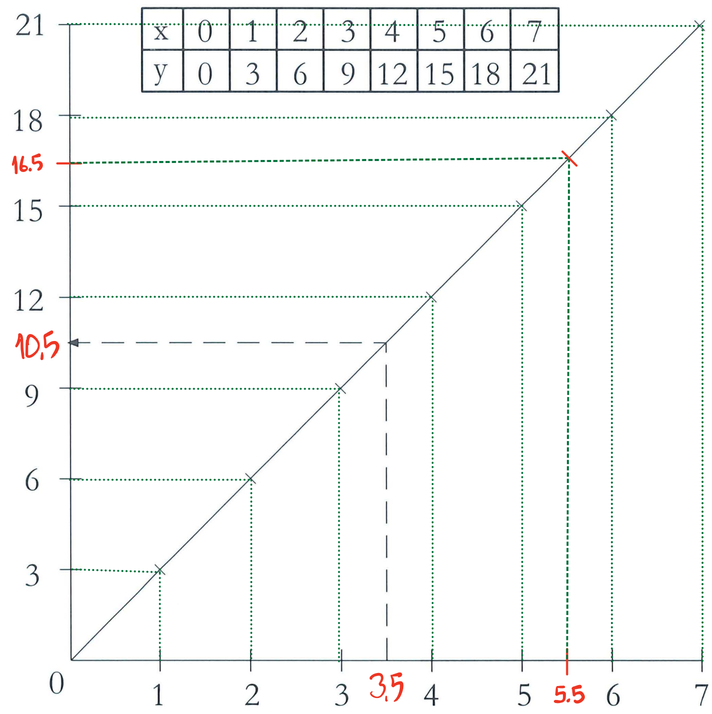
Day	1	2	3	4	5	6	7
Temp (°C)	22	20	26	24	30	28	26

The points on the graph are plotted using the coordinates in the table. The first point has the coordinates (1,22), the second (2,20) and so on. When the points have been plotted, they are joined together by straight lines.

Example

The following table gives corresponding values of x and y. Plot a graph of this information.

x	0	1	2	3	4	5	6	7
y	0	3	6	9	12	15	18	21



When the points are plotted and joined a straight line is obtained. When a graph is a straight line or a smooth curve then it can be used to find values that are not present in the given table.

For example, what is the value of y when x = 3.5?

To do this draw a vertical line from 3.5 on the x-axis up to the line and then draw a horizontal line from this point to the y-axis and read off the value (i.e. in this case 10.5).

So when x = 3.5, y = 10.5

Finding values in this way is called ***interpolation***.

Graphs of Linear Equations

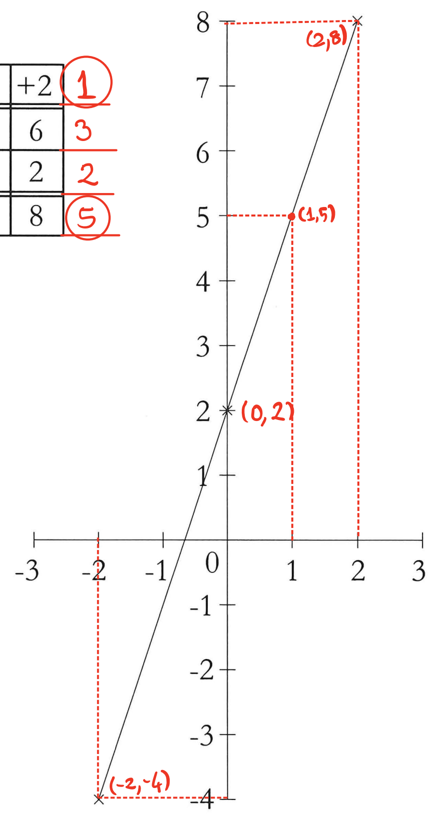
Equations of the type y = 3x + 2, where the highest powers of x and y present are the first i.e. (no y^2, y^3, x^2, x^3) are called ***linear equations***. The reason for this is that all equations of this type give rise to graphs that are straight lines.

When drawing graphs of this type, technically only two points are required but it is wiser to take three points as this third point acts as a check on the other two.

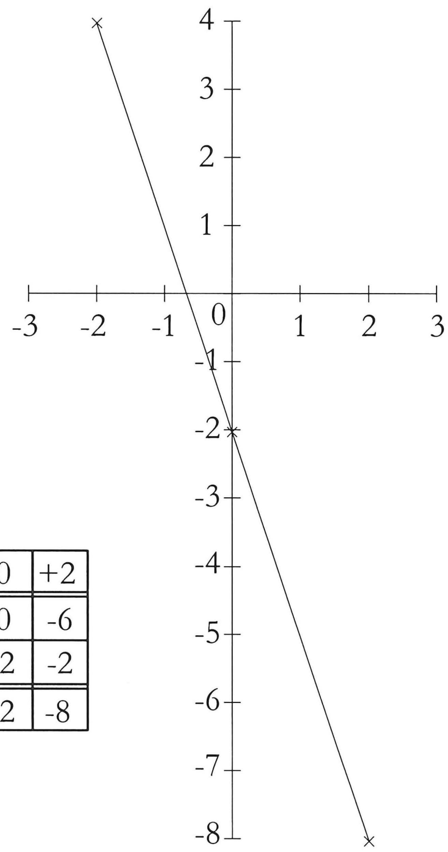
Example

Draw the graph of y = 3x + 2, taking values of x between -2 and +2.

x	-2	0	+2	1
3x	-6	0	6	3
+2	2	2	2	2
y	-4	2	8	5



If we now draw the graph of y = -3x - 2 taking values of x between - 2 and +2.



x	-2	0	+2
-3x	6	0	-6
-2	-2	-2	-2
y	4	-2	-8

If we now, compare the two equations of the graphs.

i.e. $y = 3x + 2$

$y = -3x - 2$

The only things which have changed are the signs. The graphs however are completely different.

Changing the sign in front of the 3x has made the graph run from top left to bottom right instead of from bottom left to top right. Changing the sign in front of the 2 has made the graph cut the y-axis below the origin instead of above it. If we look at the individual graphs, the first one cuts the y-axis at +2 and the second at – 2. This information is shown in the respective equations:

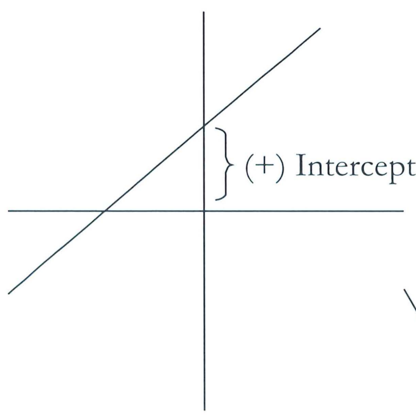
i.e. $y = 3x + 2$ and $y = -3x - 2$

And so the non-x term on the right hand side of the equation tells us exactly where the line cuts the y-axis. It has a special name, it is a called the ***intercept***.

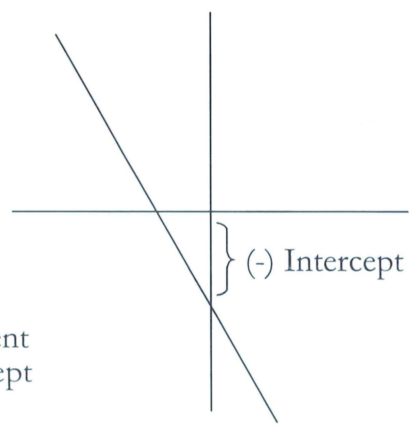
The number in front of the x-term tells us the value of the ***gradient*** (or slope). The -3 tells us that the value of the gradient is ***negative***. The +3 tells us that the value of the gradient is ***positive***.

i.e. $y = 3x + 2$ and $y = -3x - 2$

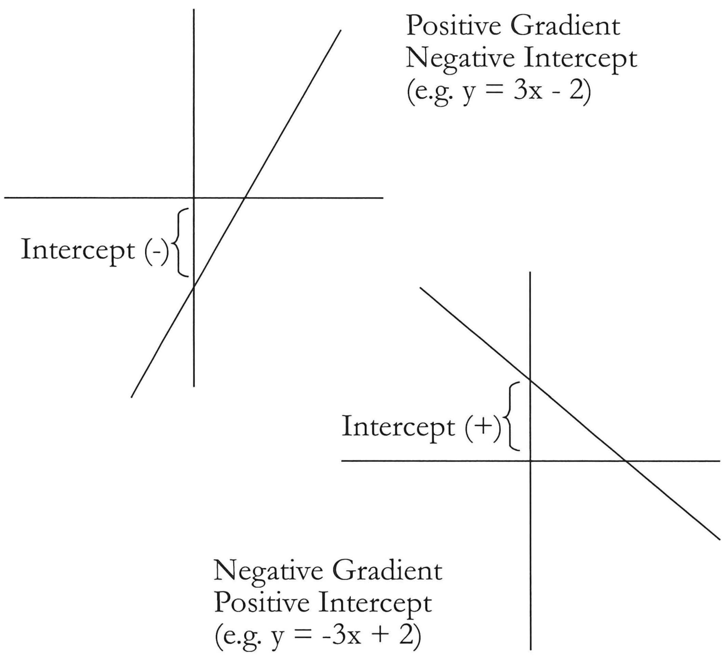
So straight lines that have a positive gradient run from bottom left to top right and those with a negative gradient from top left to bottom right.



Positive Gradient
Positive Intercept
(e.g. $y = 3x + 2$)



Negative Gradient
Negative Intercept
(e.g. $y = -3x - 2$)

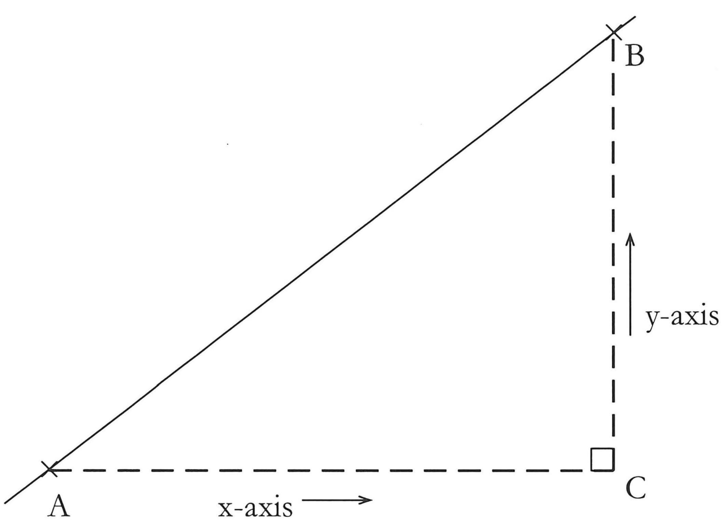


Method to Find the Value of the Gradient from a Straight-Line Graph

The gradient (or slope) of a straight line is a measure of how steep it is.

The method to find its value is:-

- Choose two points on the line, the further apart the better.
- Complete a right-angled triangle where the distance between the points is the hypotenuse.



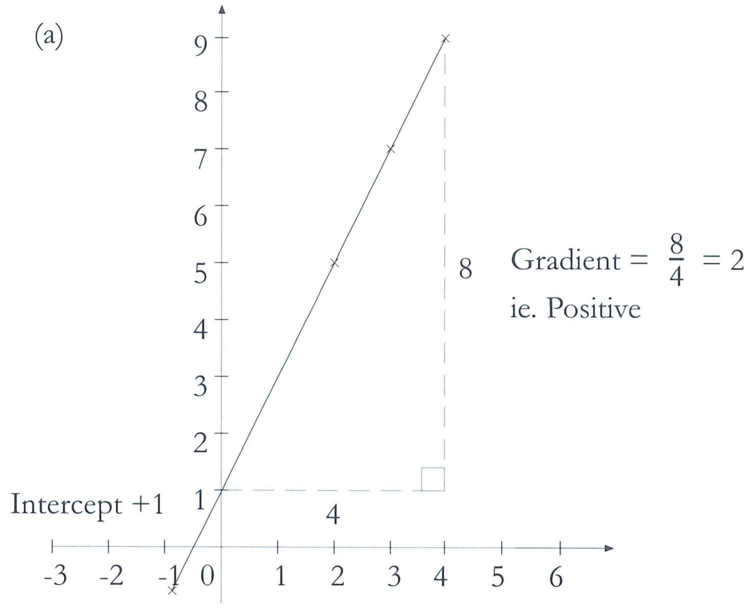
The gradient is given by:

$$\frac{BC}{AC} = \frac{\text{Difference in y - coordinates}}{\text{Difference in x - coordinates}}$$

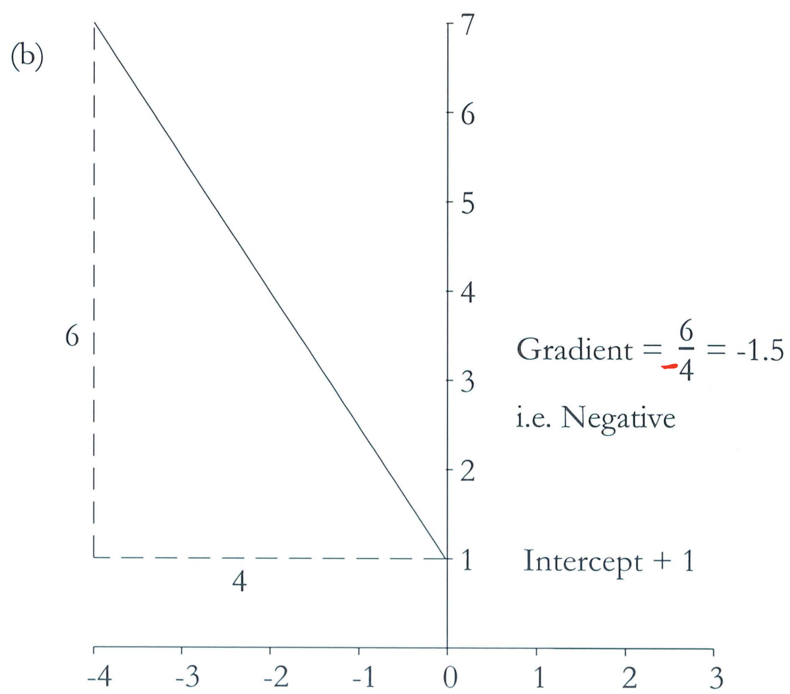
Example 1

Consider the following straight-line graphs.

(a)



(b)



In (a) the gradient is +2 and the intercept is +1. In (b) the gradient is -1.5 and the intercept is +1.

So the equation of (a) is:

$$y = 2x + 1$$

And that of (b) is:

$$y = -1.5x + 1$$

In fact every linear equation can be written in the general form:

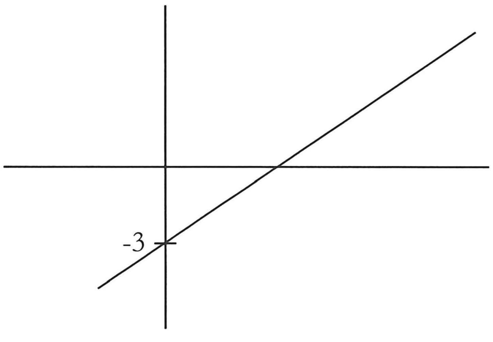
$$y = mx + c$$

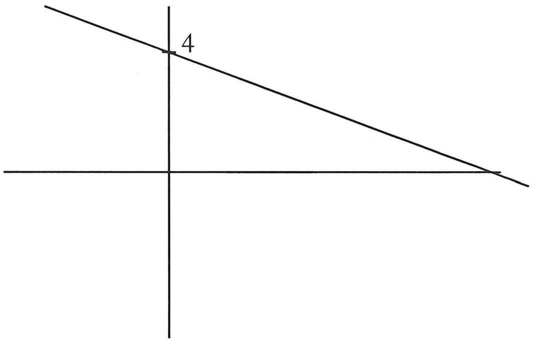
Where m is the gradient and c is the intercept. This is known as the ***general straight-line law***. This law enables us to describe graphs without actually drawing them.

For example, $y = 5x - 3$, will give a straight line of positive gradient 5 and it will cross the y-axis below the origin at -3.

$y = -3x + 4$, will give a straight line of negative gradient 3 and it will cross the y-axis above the origin at +4.

Rough sketches of these graphs would be: -





It should be noted that not all linear equations are in the standard form $y = mx + c$ to begin with. In these cases, the equation must be rearranged.

Example 2

$y = 5 - 2x$ (change the position of the terms on the right hand side)

$y = -2x + 5$ (therefore: $m = -2$ and $c = 5$).

In the case of $3x + 5y = 8$ then rearranging to $5y = -3x + 8$ and dividing both sides by 5, gives:

$y = -\frac{3}{5}x + \frac{8}{5}$

$$y = -\frac{3}{5}x + \frac{8}{5} \text{ (therefore: } m = -\frac{3}{5} \text{ or } -0.6$$

$$c = \frac{8}{5} \text{ or } 1.6)$$

The values of m and/or c can be calculated without the actual equation or graph provided we are given relevant information. We can then write down the equation.

Example 3

Find the values of m and c if the straight-line **$y = mx + c$** passes through the point (2,3) and has a gradient of 4.

x,y

$y=4x+c$

$3=4*2+c \Rightarrow \boxed{c=-5}$

$$y = mx + c$$

If the graph passes through (2,3) this tells us that x = 2 when y = 3. Substituting this information in $y = mx + c$ we get:

$$3 = m(2) + c$$

$$3 = 2m + c$$

Now if $m = 4$

$$3 = 2(4) + c$$

i.e. $3 = 8 + c$

$$c = 3 - 8$$

$$c = -5$$

The equation of the line is: $y = 4x - 5$

Example 4

Find the values of m and c if the straight-line **$y = mx + c$** passes through the point (4,3) and the intercept on the y-axis is – 3.

x,y

$y=mx-3$

$3=m4-3 \Rightarrow m=\frac{6}{4}=1.5$

When x = 4, y = 3. Substituting these values in the equation we get:-

$$3 = m(4) + c$$

$\boxed{y=1.5x-3}$

$$3 = 4m + c$$

Now if $c = -3$,

then $3 = 4m - 3$

i.e. $4m = 3 + 3$

$4m = 6$

i.e. $m = 1.5$

$\therefore$ The equation of the line is:

$$y = 1.5x - 3$$

Example 5

Find the values of m and c if the straight-line **$y = mx + c$** passes through the points (2,8) and (4,14).

x_1, y_1 $\quad$ x_2, y_2

$y_1 = mx_1 + c$ | $8 = 2m + c$ $\quad$ /-1

$y_2 = mx_2 + c$ | $14 = 4m + c$

$6 = 2m \Rightarrow \boxed{m=3}$

$8 = 2*3 + c \Rightarrow \boxed{c=2}$

$14 = 4*3 + c \Rightarrow \boxed{c=2}$

$$y = mx + c$$

Using this information we can form two simultaneous equations:

i.e. when $x = 2$, $y = 8$, $\therefore$ $8 = 2m + c$ (1)

and when $x = 4$, $y = 14$, $\therefore$ $14 = 4m + c$ (2)

Subtract (1) from (2)

$14 = 4m + c$ (2)

$8 = 2m + c$ (1)

$6 = 2m$

$m = 3$

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$

$m = \frac{14-8}{4-2} = \frac{6}{2} = 3$

$y = 3x + c$

$y_1 = 3x_1 + c$

$8 = 3*2 + c \Rightarrow \boxed{c = 2}$

If we now substitute for m in equation (1) we have:

$8 = 2(3) + c$

$8 = 6 + c$

$c = 2$

Example! Find the values of m and c passing through the points $(-2, 3)$ $(5, -1)$

$x_1 \ y_1 \quad x_2 \ y_2$

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-3}{5-(-2)} = \frac{-4}{7}$

$\therefore$ The equation of the line is:

$$y = 3x + 2$$

$y = -\frac{4}{7}x + c$

$y_2 = -\frac{4}{7}x_2 + c$

$-1 = -\frac{4}{7} * 5 + c$

$c = \frac{20}{7} - 1 = \frac{13}{7}$

$y = -\frac{4}{7}x + \frac{13}{7}$

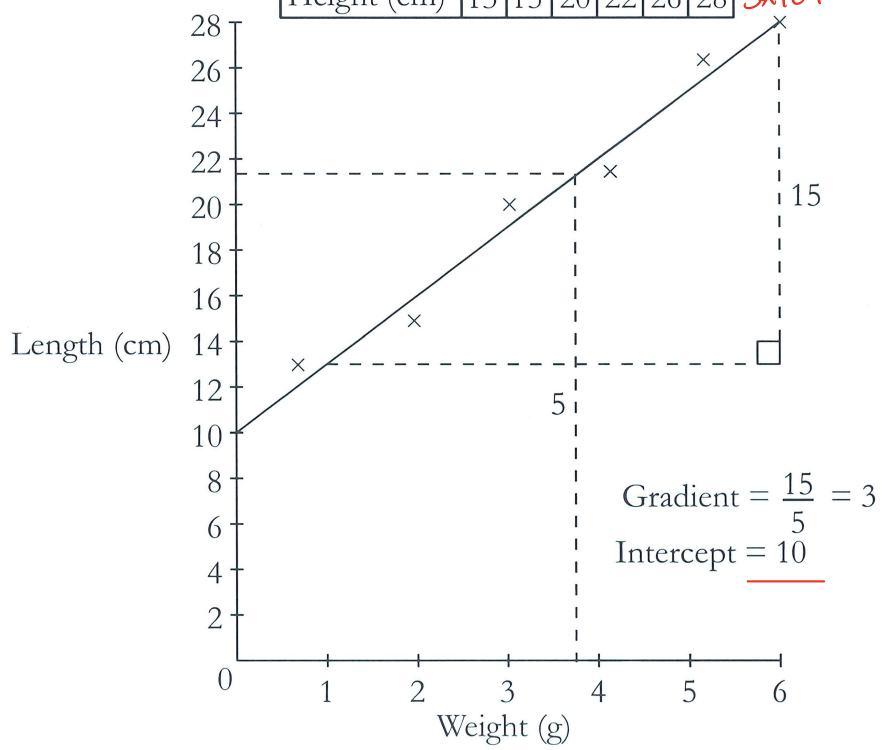
Experimental Data

An important application of the straight-line law is the determination of an equation relating two quantities obtained from experiment.

Example

In an experiment, a researcher hung weights onto a standard piece of rubber to see how much it stretched with increasing load. The increases in length were measured and recorded in the following table. A graph was then drawn.

Weight (g)	1	2	3	4	5	6	10 g
Height (cm)	13	15	20	22	26	28	$3\times10+10=40$ cm



H=3W+10

As can be seen not all the points are on the line. This is because measurements from experiments can be inaccurate. They depend very much on the accuracy of instruments/machinery used. So we must allow for experimental error.

In such cases the best line is drawn (i.e.: one which best fits the points).

From the graph drawn, the value of the gradient was 3 and that of the intercept 10.

The equation or law connecting the two variables is formed by substituting these values in the **y = mx + c** equation or in this case the **L = mW + c** equation since W replaces x and L replaces y.

So we have: $L = 3W + 10$

If we now use this equation to find the extension caused by a weight not plotted on the graph (e.g.: 3.8 gm) we get:

$$L = 3(3.8) + 10$$

$$L = 21.4\text{cm}$$

The graph should give the same value or very close to it.

For example, draw a vertical line as shown from 3.8 up to the graph and from this point a horizontal line to the L. axis. The value is about 21.5.

So it is fair to say that L and W are connected by an equation of the type

$$L = 3W + 10$$

Curved Graphs

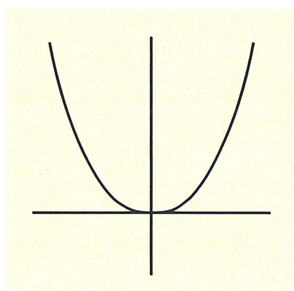
It is useful to know the general shape of the more common curves.

Quadratic Curves

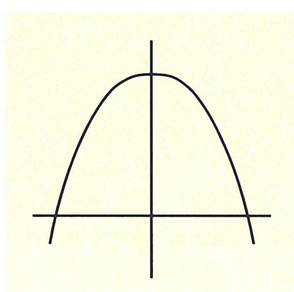
The equations of this type have an x^2 term as the highest power of x.

Examples, $x^2 - x + 8$, $2x^2 + 3x - 5$

When the x^2 term is positive the general shape of the curve is:



When the x^2 term is negative e.g. $3 + 5x - x^2$ the general shape of the curve is inverted:



These are called parabolas.

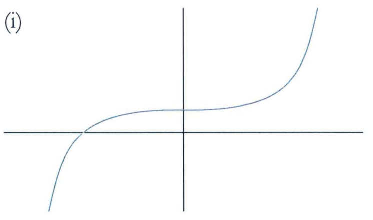
Cubic Curves

The equations of this type have an x^3 term as the highest power of x.

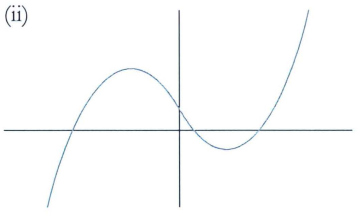
Examples, $x^3 + 5x - 3$ $3x^3 + 2x + 3$

When the x^3 term is ***positive*** the general shape of the curve can be as (i) or (ii) below:

(i)

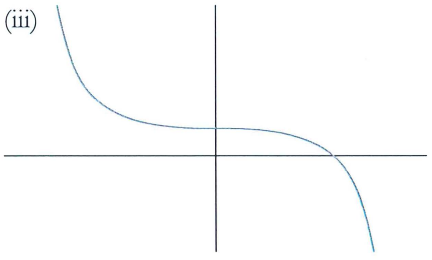


(ii)

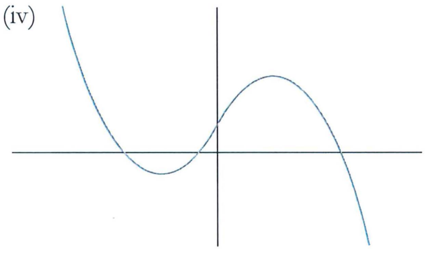


When the x^3 term is ***negative*** the general shape of the curve can be as (iii) or (iv) below:

(iii)



(iv)

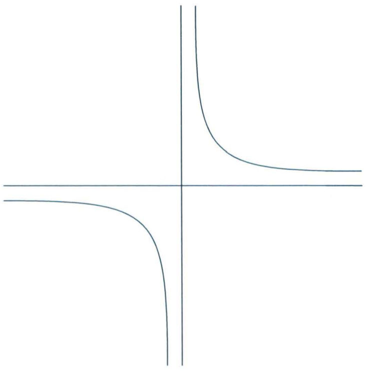


Reciprocal Curves

The equations of this type have a $\frac{1}{x}$ term present.

Examples, $y = \frac{10}{x}$ or $y = \frac{5}{x} + 3$

The general shape of the curve is:



It should be noted that the reciprocal of a negative number is a negative number, whilst the reciprocal of a positive number is a positive number.

It is not possible to find the reciprocal of zero since we cannot divide by zero. For this reason the curve has a break at x = 0 and as a result the graph consists of two separate branches.

The x-axis and y-axis are called ***asymptotes*** to the curve. The curve gets close to but never touches the asymptotes.

Graphs of Quadratic Functions

The general expression for a quadratic function of x is $ax^2 + bx + c$ where a, b and c are constants. The graphs of quadratic expressions give rise to a smooth curve called a ***parabola***.

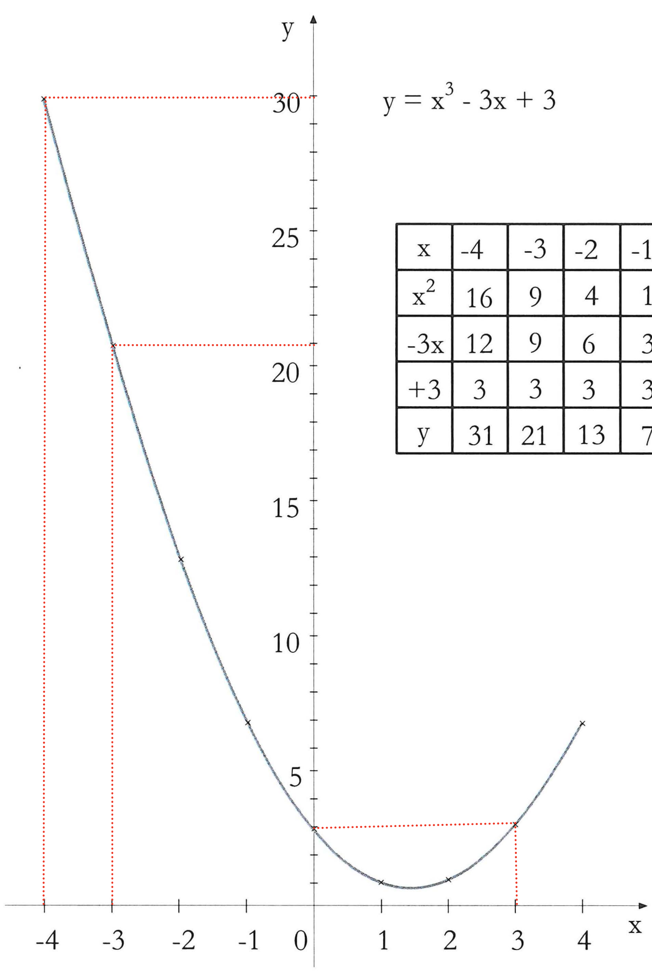
Example 1

Plot the graph of $y = x^2 - 3x + 3$ between $x = -4$ and $x = 4$.

The first step is to draw up a table as shown:

The graph is as expected a smooth curve. This is shown below.

The range of x is –4 to +4 (i.e. 8). The range of y is 1 to 31 (i.e. 30)



x	-4	-3	-2	-1	0	1	2	3	4
x^2	16	9	4	1	0	1	4	9	16
-3x	12	9	6	3	0	-3	-6	-9	-12
+3	3	3	3	3	3	3	3	3	3
y	31	21	13	7	3	1	1	3	7

Example 2

Draw up a table of values for $y = -2x^2 - 3x - 6$ for values of x between –4 and +4 and draw the corresponding graph. Alternatively we could calculate each value of y from its corresponding x-value by substituting for x in the equation of the graph. For example:

When $x = 2$ $\quad y = -2(2)^2 - 3(2) - 6$

$= -8 - 6 - 6$

$= -20$

The other values of y are calculated in a similar fashion. The graph is to be drawn on standard graph paper that is 18cm wide and 28cm high.

Since the range for x is –4 to +4 (i.e. 8), the scale on the x-axis is 2cm = 1 unit.

Since the range for y is –5 to 50 (45), the scale on the y-axis is 2cm = 5 units.

The graph is once again as expected a smooth curve but this type is inverted because of the negative x^2 term.

x	-4	-3	-2	-1	0	1	2	3	4
$-2x^2$	-32	-18	-8	-2	0	-2	-8	-18	-32
-3x	12	9	6	3	0	-3	-6	-9	-12
-6	-6	-6	-6	-6	-6	-6	-6	-6	-6
y	-26	-15	-8	-5	-6	-11	-20	-33	-50