Prime Numbers

A prime number is a number that can only be divided by 1 or itself.

The first prime number is 2 and this is the only even prime number. All other prime numbers are odd but not all odd numbers are prime numbers.

2, 3, 5, 7, ~~9~~, 11, 13, ~~15~~, 17, 19, ~~21~~, 23, ~~25~~, ~~27~~, 29,
31, ~~33~~, ~~35~~, 37, ~~39~~, 41, 43, ~~45~~, 47, ~~49~~.

Some odd numbers are crossed out because besides being divisible by 1 they are also divisible by at least 3, 5 or 7 so they are not prime numbers. The complete list of prime numbers up to 50 is:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 39, 41, 43 and 47

Prime Factor

80	2
40	2
20	2
10	2
5	5
1	

As we have seen previously, 5 is a factor of 80. However since 5 is a prime number then 5 is also called a prime factor of 80.

So a prime factor of a number is a factor of that number which is also a prime number.

2 and 5 the prime factors of 80

Method to find the Prime Factors of a Number

Start with the lowest prime number that will divide into the number under investigation and do this repetitively until it will not divide exactly. Once this happens, move up to the next prime number and continue until the answer to the division is 1.

Example 1 Find the prime factors of 30

$$\begin{array}{r|l} 2 & 30 \\ 3 & 15 \\ 5 & 5 \\ & 1 \end{array}$$

So the prime factors of 30 are 2, 3 and 5.

Example 2 Find the prime factors of 80

So the prime factors of 80 are 2 and 5. Note that when the prime factors are asked for they are only listed once (as in this example) although there are four divisions by 2, it is only listed as a prime factor once.

$$\begin{array}{r|l} 2 & 80 \\ 2 & 40 \\ 2 & 20 \\ 2 & 10 \\ 5 & 5 \\ & 1 \end{array}$$

Note the following type of question:

What is 80 expressed as a product of its prime factors?

The answer would be: 2 × 2 × 2 × 2 × 5, because the product of just 2 and 5 would not give 80.

What is 80 expressed as a product of its prime factors in index form?

The answer would be: $2^4 \times 5$

Prime Factor method to find the HCF

Example

Find the HCF of 60 and 80

a) Write down the prime factors for 60

b) Match these from the prime factors for 80

c) Multiply the matched factors

$$\begin{array}{r|l} 2 & 60 \\ 2 & 30 \\ 3 & 15 \\ 5 & 5 \\ & 1 \end{array} \qquad \begin{array}{r|l} 2 & 80 \\ 2 & 40 \\ 2 & 20 \\ 2 & 10 \\ 5 & 5 \\ & 1 \end{array}$$

(a)

$2 \times 2 \times 3 \times 5$

(b)

$$\begin{array}{ccccccccccc} & & & & \boxed{2} & \times & \boxed{2} & \times & 3 & \times & \boxed{5} & (60) \\ 2 & \times & 2 & \times & \boxed{2} & \times & \boxed{2} & & & \times & \boxed{5} & (80) \end{array}$$

(c)

$2 \times 2 \times 5 = 20$

The HCF of 60 and 80 is 20.

Prime Factor method to find the LCM

Example 1

Find the LCM of 60 and 80.

The first part is the same as for finding the HCF

$$\begin{array}{r|l} 2 & 60 \\ 2 & 30 \\ 3 & 15 \\ 5 & 5 \\ & 1 \end{array} \qquad \begin{array}{r|l} 2 & 80 \\ 2 & 40 \\ 2 & 20 \\ 2 & 10 \\ 5 & 5 \\ & 1 \end{array}$$

a) Write down the factors that appear in **either** list once.

b) Now write down the **highest power** of these factors that appear in **either** list.

c) Multiply these out

a) $2 \times 3 \times 5$

b) $2^4 \times 3 \times 5$

c) $2^4 \times 3 \times 5 = 240$

The LCM of 60 and 80 is 240.

Example 2

Find the LCM of 25 and 85

$$\begin{array}{r|l} 5 & 25 \\ 5 & 5 \\ & 1 \end{array} \qquad \begin{array}{r|l} 5 & 85 \\ 17 & 17 \\ & 1 \end{array}$$

a) Write down the factors that appear in **either** list once.

b) Now write down the **highest power** of these factors, which appear in **either** list.

c) Multiply these out

a) 5×17

b) $5^2 \times 17^1$

c) $5^2 \times 17 = 425$

The LCM of 25 and 85 is 425.

Fractions

A fraction is part of a whole, where the whole can be anything we define.

A fraction consists of two parts, the numerator (top) and the denominator (bottom).

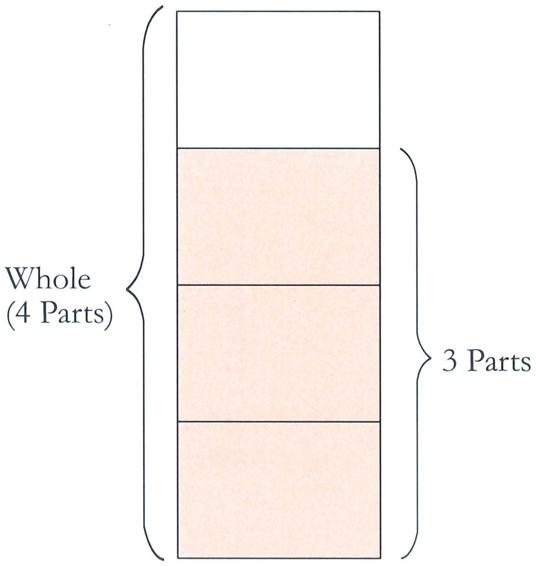
The bottom tells us how many equal parts the whole has been divided into and the top tells us how many of these equal parts are taken.

Example

Consider the fraction $3/4$

The whole has been divided into 4 equal parts and three of those parts have been shaded.

So the shaded parts represent $3/4$ of the whole.



Types of Fractions

Proper Fraction

In this type the top of the fraction is less than the bottom.

$\frac{3}{4}, \frac{2}{5}, \frac{1}{8}, \frac{17}{20}$ are all examples of ***proper fractions***.

Improper Fraction

In this type, the top of the fraction is more than the bottom.

$\frac{5}{2}, \frac{6}{5}, \frac{12}{4}, \frac{18}{11}$ are all examples of ***improper fractions***.

These types of fractions are also called ***top-heavy*** fractions.

Mixed Number

In this type you have the sum of an ***integer*** (whole number) and a proper fraction.

$3 + \frac{3}{4} = (3\frac{3}{4})$

$1\frac{2}{3}$ and $5\frac{7}{8}$ are further examples of ***mixed number fractions***.
Mixed number fractions can be changed into improper fractions and vice versa. For example,

$$5\frac{2}{3} = \frac{(3 \text{ x } 5) + 2}{3} = \frac{17}{3}$$

$$4\frac{2}{5} = \frac{(5 \text{ x } 4) + 2}{5} = \frac{22}{5}$$

If you find this difficult, the following diagram might help.

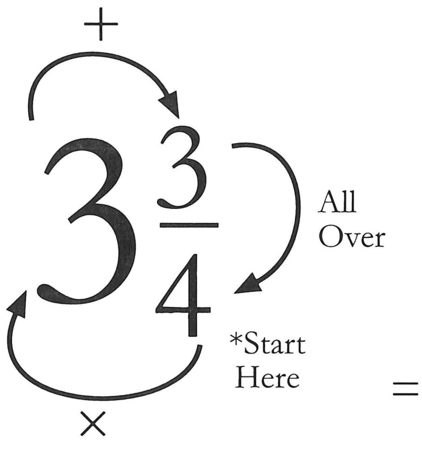
Example 1 Change $3\frac{3}{4}$ into an improper fraction

$$3\frac{3}{4} = \frac{(3\times4)+3}{4}$$
$$= \frac{15}{4}$$

Example 2:

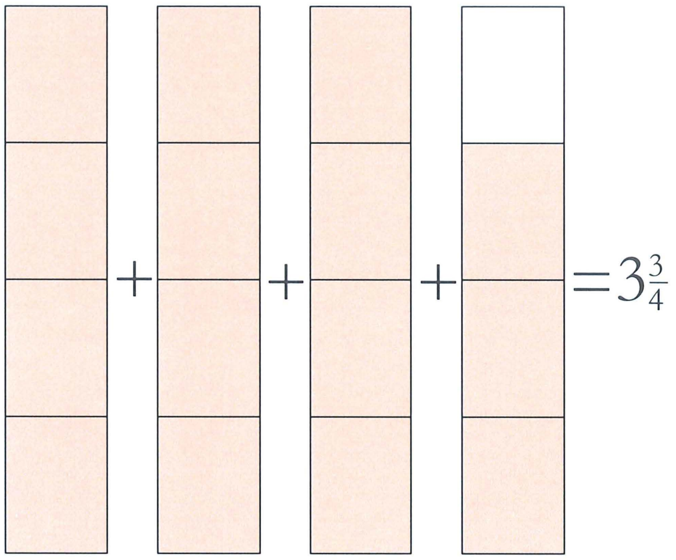
Change $7\frac{7}{13}$ into an improper fraction.

$$7\frac{7}{13} = \frac{(7\times13)+7}{13} = \frac{91+7}{13}$$
$$= \frac{98}{13}$$



$= \frac{15}{4}$

Another method is to consider what $3\frac{3}{4}$ would look like in diagrammatic form as shown on the next page.

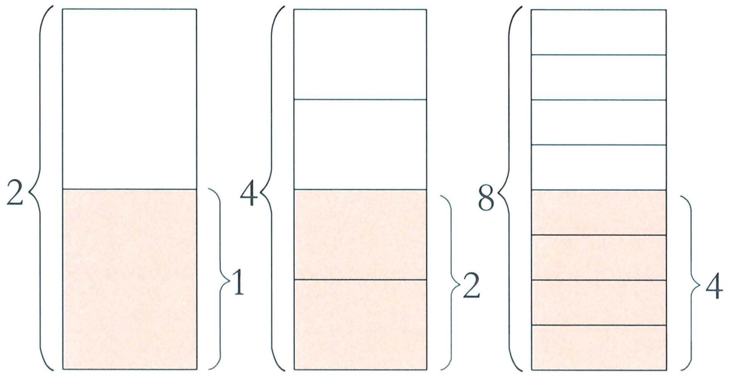


The shaded part is equal to 3¾, but this also equals fifteen quarters $= \frac{15}{4}$

In the reverse way $\frac{15}{4} = 3¾$ (15 ÷ 4 = 3 remainder 3)

Equivalent Fractions

If we consider three equal 'wholes' and divide one of them into two equal parts, one into four equal parts and one into eight equal parts.



If the same portion of the whole is shaded in each one, then the shaded parts must be equal.

For example, $\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$

These are called Equivalent Fractions. Their values are the same!

$$\frac{1}{2} = 1 \div 2 = 0.5 \qquad \frac{2}{4} = 2 \div 4 = 0.5 \qquad \frac{4}{8} = 4 \div 8 = 0.5$$

$$\frac{1}{2} \times \frac{2}{2} = \frac{1\times2}{2\times2} = \frac{2}{4} \times \frac{2}{2} = \frac{2\times2}{4\times2} = \frac{4}{8}$$

This leads to the very important conclusion that the value of a fraction remains the same if the top and bottom of it are multiplied by the same number.

$$\frac{1\ (\times 2)}{2\ (\times 2)} = \frac{2}{4}\ ,\quad \frac{2\ (\times 2)}{4\ (\times 2)} = \frac{4}{8}$$

The reverse of this is also true (i.e. the value of a fraction remains the same if the top and bottom of it are divided by the same number). This is called cancelling down.

$$\frac{20\ (\div 2)}{30\ (\div 2)} = \frac{10}{15}\ ,\quad \frac{10\ (\div 5)}{15\ (\div 5)} = \frac{2}{3}$$

In the above case the fraction ⅔ is said to be in its lowest terms because there is now no number that will divide exactly into the top and bottom of it.

Example 1 Reduce $\frac{30}{64}$ to its lowest terms

Divide 2 into the top and bottom.

$$\frac{30\ (\div 2)}{64\ (\div 2)} = \frac{15}{32}$$

There is no number that goes into ***both*** 15 and 32 exactly and so it is said to be in its lowest terms.

Example 2 Reduce $\frac{64}{96}$ to its lowest terms

$$\frac{64\,/2}{96\,/2} = \frac{32\,/2}{48\,/2} = \frac{16}{24}$$

$$\frac{16/2}{24/2} = \frac{8/2}{12/2} = \frac{4/2}{6/2} = \frac{2}{3}$$

Dividing top and bottoms by two consecutively.

$$\frac{64\ (\div 2)}{96\ (\div 2)} = \frac{32}{48}\ ,\quad \frac{32\ (\div 2)}{48\ (\div 2)} = \frac{16}{24}$$

$$\frac{16\ (\div 2)}{24\ (\div 2)} = \frac{8}{12}\ ,\quad \frac{8\ (\div 2)}{12\ (\div 2)} = \frac{4}{6}$$

$$\frac{4\ (\div 2)}{6\ (\div 2)} = \frac{2}{3}$$

Example!
Find the simplest form of
$\frac{60}{240}$

$$\frac{60\,/10}{240/10} = \frac{6\,/2}{24/2} = \frac{3/3}{12/3} = \frac{1}{4}$$

Or, if our knowledge of tables is O.K. we could have said:

$$\frac{64\ (\div 32)}{96\ (\div 32)} = \frac{2}{3}$$

But do not worry, you can reduce safely in stages if you prefer!

$$\frac{60/2}{240/2} = \frac{30\ /2}{120\ /2} = \frac{15/3}{60/3} = \frac{5/5}{20/5} = \frac{1}{4}$$

Arithmetical Operations with Fractions

Addition

Look at the fraction sum $\frac{3}{4} + \frac{5}{8}$ $= \frac{3\times2}{4\times2} + \frac{5}{8} = \frac{6}{8} + \frac{5}{8} = \frac{6+5}{8} = \frac{11}{8}$

This is like saying 3 oranges plus 5 bananas. Quarters are different objects to eighths. The reason for this is that their ***denominators*** (bottoms) are different. If we had $\frac{2}{7} + \frac{3}{7}$ then since the denominators are the same, all we have to do is to add their ***numerators*** (tops). For example, $\frac{2}{7} + \frac{3}{7} = \frac{5}{7}$

So when two fractions have different denominations they cannot be added together directly unless we first express them with the same denominators.

Example $\frac{3}{5} + \frac{1}{8}$ $= \frac{3\times8}{5\times8} + \frac{1\times5}{8\times5} = \frac{24}{40} + \frac{5}{40} = \frac{24+5}{40} = \frac{29}{40}$

Now, if we make the bottoms the same by multiplying by 8 and 5 respectively, as shown, then:-

$$\frac{3\ (\times\ 8)}{5\ (\times\ 8)} = \frac{24}{40} \quad \text{and} \quad \frac{1\ (\times\ 5)}{8\ (\times\ 5)} = \frac{5}{40}$$

The lowest common denominator in this case is the lowest common multiple of 5 and 8 i.e. 40.

We can now add the tops (numerators) together and put them over this common denominator.

$$\frac{3}{5} + \frac{1}{8} = \frac{24}{40} + \frac{5}{40} = \frac{29}{40}$$

Now

$$\frac{24}{40} + \frac{5}{40} \equiv \frac{24+5}{40}$$

So a better way of setting out the problem would be:

$$\frac{3}{5} + \frac{1}{8}$$
$$= \frac{24+5}{40}$$
$$= \frac{29}{40}$$

The Step-by-Step Process is:

- Find the lowest common denominator of the fractions to be added.
- Express both these fractions with this lowest common denominator
- Add the numerators and put over this lowest common denominator.
- If the answer is a top-heavy fraction, change to a mixed number.
- Express this fraction in its lowest terms if needed.

Examples

1) $\frac{5}{8} + \frac{3}{4} = \frac{5+6}{8} = \frac{11}{8} = 1\frac{3}{8}$

2) $\frac{3}{16} + \frac{5}{32} = \frac{6+5}{32} = \frac{11}{32}$

3) $\frac{3}{8} + \frac{1}{5} = \frac{15+8}{40} = \frac{23}{40}$

4) $\frac{5}{8} + \frac{2}{9} = \frac{45+16}{72} = \frac{61}{72}$

5) $\frac{7}{8} + \frac{5}{6} = \frac{42+40}{48} = \frac{82}{48} = 1\frac{34}{48} = 1\frac{17}{24}$

When adding mixed number fractions together the whole number part and the fraction part are added separately.

Example 1 $5\frac{2}{3} + 1\frac{5}{6}$

$(5+1) + \left(\frac{2}{3} + \frac{5}{6}\right)$

$6\left(\frac{4}{6} + \frac{5}{6}\right)$

$\frac{9}{6} = \frac{6}{6} + \frac{3}{6}$

$= 1\frac{3}{6}$

$6\frac{9}{6} = 6\left[1\frac{3}{6}\right]$

$= 7\frac{3}{6}$

$= 7\frac{1}{2}$

- Find the lowest common denominator of the fractions to be added.
- Express both these fractions with this lowest common denominator.
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- If the answer is a top-heavy fraction, change to a mixed number.
- Express this fraction in its lowest term if needed.

$$5\frac{2}{3} + 1\frac{5}{6}$$

$$= 5 + \frac{2}{3} + 1 + \frac{5}{6}$$

$$= 6 + \frac{4+5}{6}$$

$$= 6 + \frac{9}{6}$$

$$= 6 + 1\frac{3}{6}$$

$$= 6 + 1\frac{1}{2}$$

$$= 7\frac{1}{2}$$

Example 2 $2\frac{5}{8}+3\frac{3}{4}$ $=5\left(\frac{5}{8}+\frac{3}{4}\right)=5\left(\frac{5}{8}+\frac{6}{8}\right)=5\,\frac{11}{8}$

$6\frac{3}{8}$ $=5\left(\underset{1}{\frac{8}{8}}+\frac{3}{8}\right)$

- Find the lowest common denominator of the fractions to be added.
- Express both these fractions with this lowest common denominator.
- Add the numerators and put over this lowest common denominator.
- If the answer is a top-heavy fraction, change to a mixed number.
- Express this fraction in its lowest terms if needed.

$$
\begin{aligned}
&2\frac{5}{8}+3\frac{3}{4}\\
&= 2+\frac{5}{8}+3+\frac{3}{4}\\
&= 5+\frac{5}{8}+\frac{3}{4}\\
&= 5+\frac{5+6}{8}\\
&= 5+\frac{11}{8}\\
&= 5+1\frac{3}{8}\\
&= 6\,\frac{3}{8}
\end{aligned}
$$

Subtraction

The method is similar to addition except that in the final stages the numerators are subtracted.

Example

$\frac{3}{4}\times\frac{5}{5}-\frac{1}{5}\times\frac{4}{4}$ ↤

$\frac{15}{20}-\frac{4}{20}=\frac{15-4}{20}$

$=\frac{11}{20}$

1) $\frac{3}{4}-\frac{1}{5}=\frac{15-4}{20}=\frac{11}{20}$

2) $\frac{2}{3}-\frac{1}{6}=\frac{4-1}{6}=\frac{3}{6}=\frac{1}{2}$

$\frac{2}{3}-\frac{1}{6}=\frac{2}{3}\times\frac{2}{2}-\frac{1}{6}=\frac{4}{6}-\frac{1}{6}=\frac{3/3}{6/3}$

$=\frac{1}{2}=0.5$

When subtracting mixed number fractions it is better to change them into top-heavy fractions first. The following example shows why.

Example: $\frac{7}{12}-\frac{6}{11}=\frac{7}{12}\times\frac{11}{11}-\frac{6}{11}\times\frac{12}{12}=\frac{77}{132}-\frac{72}{132}=\frac{77-72}{132}=\frac{5}{132}$

Example: $\underbrace{\frac{9}{10}-\frac{8}{15}}-\frac{3}{5}\Longrightarrow\frac{9}{10}-\frac{8}{15}=\frac{9}{10}\times\frac{3}{3}-\frac{8}{15}\times\frac{2}{2}=\frac{27}{30}-\frac{16}{30}=\frac{27-16}{30}=\frac{11}{30}$

$\frac{11}{30}-\frac{3}{5}=\frac{11}{30}-\frac{3}{5}\times\frac{6}{6}=\frac{11}{30}-\frac{18}{30}=\frac{11-18}{30}=\frac{-7}{30}$

Example: $\underbrace{\frac{12}{13}-\frac{7}{39}}+\frac{2}{65}\Longrightarrow\frac{12}{13}-\frac{7}{39}=\frac{12}{13}\times\frac{3}{3}-\frac{7}{39}=\frac{36}{39}-\frac{7}{39}=\frac{36-7}{39}=\frac{29}{39}$

$\boxed{\frac{29}{39}}+\frac{2}{65}=\frac{29}{39}\times\frac{5}{5}+\frac{2}{65}\times\frac{3}{3}=\frac{145}{195}+\frac{6}{195}=\frac{151}{195}$

Division of Fractional Terms

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{a\times d}{b\times c}$$

Method A	Method B
$7\frac{3}{4} - 5\frac{1}{3}$	$7\frac{3}{4} - 5\frac{1}{3}$
$= 7 + \frac{3}{4} - (5 + \frac{1}{3})$	$= \frac{31}{4} - \frac{16}{3}$
$= 7 + \frac{3}{4} - 5 - \frac{1}{3}$	$= \frac{93 - 64}{12}$
$= 7 - 5 + \frac{3}{4} - \frac{1}{3}$	$= \frac{29}{12}$
$= 2 + \left(\frac{9}{12} - \frac{4}{12}\right)$	$= 2\frac{5}{12}$
$= 2\frac{5}{12}$	

Method B is less complicated and less prone to errors!

Example 1 $5\frac{7}{16} - 2\frac{2}{3}$

$$5\frac{7}{16} - 2\frac{2}{3}$$

$$= \frac{87}{16} - \frac{8}{3}$$

$$= \frac{261 - 128}{48}$$

$$= \frac{133}{48}$$

$$= 2\frac{37}{48}$$

$$= \frac{5\times16+7}{16} - \frac{2\times3+2}{3}$$

$$= \frac{87}{16} - \frac{8}{3}$$

$$= \frac{87}{16}\times\frac{3}{3} - \frac{8}{3}\times\frac{16}{16}$$

$$= \frac{261}{48} - \frac{128}{48} = \frac{261-128}{48} = \frac{133}{48}$$

$$= 2\frac{37}{48}$$

$$\begin{array}{r} 2 \\ 48\overline{)133} \\ -96 \\ \hline 37 \end{array}$$

Multiplication of Fractions

Example

Multiply the ***numerators*** (tops) together.

Multiply the ***denominators*** (bottoms) together.

Express numerator answer over the denominator answer.

Cancel down to lowest terms (if needed).

$$\frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$$

Further Examples

$$\frac{2}{3} \times \frac{5}{7} = \frac{10}{21}$$

$$\frac{20}{56} = \frac{20/2}{56/2} = \frac{10/2}{28/2} = \frac{5}{14}$$

$$\frac{5}{8} \times \frac{4}{7} = \frac{20}{56} \quad \text{or} \quad \frac{5}{14}$$ (ie dividing top and bottom by 4)

When multiplying mixed numbers they must be converted into top-heavy fractions first.

$$1\frac{3}{4} = \frac{1\times4+3}{4}$$
$$= \frac{7}{4}$$

$$2\frac{2}{3} = \frac{2\times3+2}{3}$$
$$= \frac{8}{3}$$

$$1\frac{3}{4} \times 2\frac{2}{3}$$
$$= \frac{7}{4} \times \frac{8}{3}$$
$$= \frac{56}{12} = \frac{56/2}{12/2} = \frac{28/2}{6/2} = \frac{14}{3}$$
$$= 4\cancel{\frac{8}{12}}\left(\frac{2}{3}\right)$$
$$= 4\frac{2}{3}$$

$$\begin{array}{r} 4 \\ 3\overline{)14} \\ -12 \\ \hline 2 \end{array}$$

$$4\frac{2}{3}$$

$$2\frac{2}{5} \times 1\frac{3}{4} = \frac{2\times5+2}{5} \times \frac{1\times4+3}{4}$$
$$= \frac{12}{5} \times \frac{7}{4} = \frac{84}{20}$$
$$= \frac{84/2}{20/2} = \frac{42}{10} = \frac{42/2}{10/2}$$
$$= \frac{21}{5}$$

$$= \frac{\overset{3}{\cancel{12}}}{5} \times \frac{7}{\underset{1}{\cancel{4}}}$$
$$= \frac{3}{5} \times \frac{7}{1}$$
$$= \frac{21}{5}$$
$$= 4\frac{1}{5}$$

$$\begin{array}{r} 4 \\ 5\overline{)21} \\ -20 \\ \hline 1 \end{array}$$

$$4\frac{1}{5}$$

Division of Fractions

This is a very similar method to multiplication except that the ÷ sign is changed to an × sign and the fraction you are dividing by is turned upside down. (i.e. inverted).

Example 1

$$3\frac{3}{4} \div 1\frac{2}{5} = \frac{3\times4+3}{4} \div \frac{1\times5+2}{5}$$
$$= \frac{15}{4} \div \frac{7}{5} = \frac{15}{4} \times \frac{5}{7}$$
$$= \frac{75}{28}$$

$$= \frac{15}{4} \div \frac{7}{5}$$
$$= \frac{15}{4} \times \frac{5}{7}$$
$$= \frac{75}{28}$$
$$= 2\frac{19}{28}$$

$$\begin{array}{r} 2 \\ 28\overline{)75} \\ -56 \\ \hline 19 \end{array}$$

$$2\frac{19}{28}$$

Example 2

$$2\frac{2}{5} \div \frac{3}{10} = \frac{2\times5+2}{5} \div \frac{3}{10} = \frac{12}{5} \times \frac{10}{3}$$

$$= \frac{4\times10}{5}$$

$$= 4\times2=8$$

$$= \frac{12}{5} \div \frac{3}{10}$$

$$= \boxed{\frac{\cancel{12}^{4}}{\cancel{5}_{1}} \times \frac{\cancel{10}^{2}}{\cancel{3}_{1}}} \quad \text{or} \quad \frac{4}{1} \times \frac{2}{1}$$

$$= 8$$

Example 3

$$1\frac{3}{4} \div 2\frac{5}{8} = \frac{1\times4+3}{4} \div \frac{2\times8+5}{8} = \frac{7}{4} \div \frac{21}{8}$$

$$= \frac{7}{4} \times \frac{8}{21}$$

$$= \frac{1\times2}{3} = \frac{2}{3}$$

$$= \frac{7}{4} \div \frac{21}{8}$$

$$= \boxed{\frac{\cancel{7}^{1}}{\cancel{4}_{1}} \times \frac{\cancel{8}^{2}}{\cancel{21}_{3}}} \quad \text{or} \quad \frac{1}{1} \times \frac{2}{3}$$

$$= \frac{2}{3}$$

NB: You may cancel down by dividing the ***top*** and ***bottom*** of a single fraction by the same number.

e.g.

$$\frac{12 \div 4}{20 \div 4} = \frac{3}{5}$$

Or you may divide the ***top*** and ***bottom*** of adjacent fractions by the same number.

e.g.

$$\frac{\cancel{7}^{1}}{\cancel{4}_{1}} \times \frac{\cancel{8}^{2}}{\cancel{21}_{3}} = \frac{2}{3}$$

Conversion of Fractions to Decimal

$\frac{12}{3}$ means 12 ÷ 3 which equals 4.

In the same way the fraction $\frac{3}{8}$ means 3 ÷ 8

i.e.

$$\begin{array}{r} 0{\cdot}375 \\ 8\,\overline{)\,3{\cdot}000} \\ \underline{24}\downarrow \\ 60 \\ \underline{56}\downarrow \\ 40 \\ \underline{40} \\ \underline{\cdot\,\cdot} \end{array}$$

We say that 0.375 is the decimal fraction that is equivalent to ³⁄₈.

In the same way:

$$3 + \frac{3}{4} = 3\frac{3}{4}$$

$$3 + 0.75 = \begin{array}{r} 3.00 \\ +\,0.75 \\ \hline 3.75 \end{array}$$

$$3\tfrac{3}{4} \longrightarrow 3 + \longrightarrow \begin{array}{r} 0{\cdot}75 \\ 4\,\overline{)\,3{\cdot}00} \\ \underline{28}\downarrow \\ 20 \\ \underline{20} \\ \underline{\cdot\,\cdot} \end{array} \longrightarrow 3.75$$

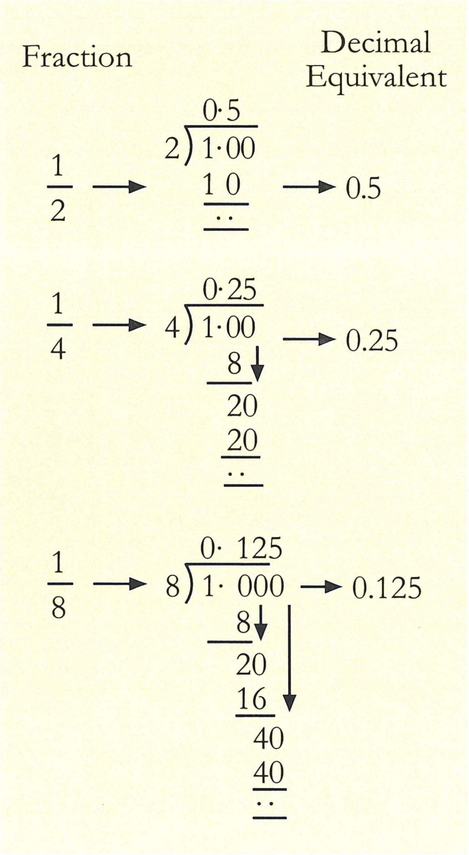
In the case of mixed number fractions, the whole number part remains the same. The fractional part is converted however, as above.

So,

2 ½ → 2.5

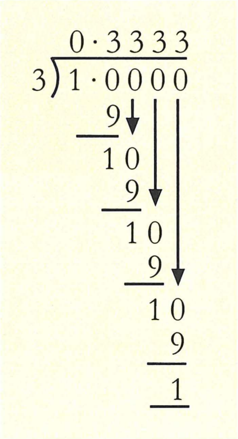
3 ¼ → 3.25

4 ⅛ → 4.125



All the above are called ***terminating decimals*** (i.e. they stop at some stage).

However, if we consider the fraction ⅓ we have:



We could continue this division forever and still not be able to arrive at an answer. When the remainder is one, it merely keeps repeating itself in a sort of "loop".

3 into 10 goes 3 remainder 1

Bring down the 0

3 into 10 goes 3 remainder 1

Bring down the 0

3 into 10 goes 3 remainder 1 ……

and so on.

This kind of decimal is called a ***recurring decimal*** and rather than write a chain of 3's we write:

$\frac{1}{3} = 0.\dot{3} \rightarrow 0.33333\cdots = 0.\dot{3} = 0.\overline{3}$

(The dot above the 3 means that the 3 recurs.)

$\frac{2}{3} = 0.\dot{6}$ (The dot above the 6 means that the 6 recurs).

Of course we can quote such decimals to so many decimal places or significant figures: -

$$\begin{array}{r} 0.45454545 \\ 11\overline{)50} \\ 44 \\ \hline 60 \\ 55 \\ \hline 50 \\ 44 \\ \hline 60 \end{array}$$

$\frac{1}{3} = 0.\dot{3} = 0.33$ (2 decimal places)

$0.6666\cdots = 0.67$

$\frac{2}{3} = 0.\dot{6} = 0.67$ (2 decimal places)

$\frac{5}{11} = 0.\dot{4}\dot{5} = 0.455$ (3 significant figures)

$\frac{5}{11} = 0.454545 = 0.\dot{4}\dot{5} = 0.\overline{45}$

Decimal places and significant figures are explained later in the chapter.

Conversion of Decimals to Fractions

$0.5 = \frac{5}{10}$ $0,51 = \frac{51}{100}$

Consider the decimal 0.375. Another way of writing this would be: $\frac{0.375}{1}$

From our work on fractions, we know that we can multiply the top of a fraction by any number we choose provided we multiply the bottom of the fraction by exactly the same number.

So: -

$$\frac{0.375\ (\times\ 1000)}{1\ (\times\ 1000)} = \frac{375}{1000} = \frac{375/5}{1000/5} = \frac{75/5}{200/5} = \frac{15/5}{40/5} = \frac{3}{8}$$

We then cancel down the fraction to the lowest terms.

For example:

$$\frac{375\ (\div\ 5)}{1000\ (\div\ 5)} = \frac{75}{200}\ ,\ \frac{75\ (\div\ 25)}{200\ (\div\ 25)} = \frac{3}{8}$$

Example 1 Convert 0.3125 to a fraction in its lowest terms

$$\frac{0.3125\ (\text{x } 10{,}000)}{1\ (\text{x } 10{,}000)} = \frac{3125}{10000}$$

$$\frac{3125\ (\div 25)}{10000\ (\div 25)} = \frac{125}{400}$$

$$\frac{125\ (\div 25)}{400\ (\div 25)} = \frac{5}{80}$$

$$\frac{3125/5}{10000/5} = \frac{625}{2000}$$

$$\frac{625/5}{2000/5} = \frac{125}{400} = \frac{125/5}{400/5} = \frac{25/5}{40/5} = \frac{5}{8}$$

Example 2 Convert 0.45 to a fraction in its lowest terms

$$\frac{0.45\ (\text{x } 100)}{1\ (\text{x } 100)} = \frac{45}{100}\ ,\ \frac{45\ (\div 5)}{100\ (\div 5)} = \frac{9}{20}$$

So remember:

$0.3125 \times 10 = 3.125$
$0.3125 \times 100 = 31.25$

To multiply by 10 move the decimal point ***one place*** to the ***right***

To multiply by 100 move the decimal point ***two places*** to the ***right***

To multiply by 1000 move the decimal point ***three places*** to the ***right***

$0.3125 \times 1000 = 312.5$
$0.12 \times 1000 = 120$
$0.120 = 0.12 = 0.1200 = 0.12000$
$0.12 \times 1000 = 0.120 \times 1000 = 120$

Approximations

The display on most calculators can accommodate up to 10 digits. In most cases this is clearly too many digits. Therefore we have to have a means of arriving at an approximate answer containing fewer digits. However, answers must have a sensible degree of accuracy and this depends upon what we are calculating.

There are three methods commonly used for such approximations.

1. Rounding a number to nearest 10, 100, 1000. → For integers
2. Rounding to a certain number of decimal places. } For decimal numbers
3. Rounding to a certain number of significant figures.

Rounding Off Numbers

Calculators in good working order do not normally make mistakes but the user could, if for example he/she presses the wrong buttons or tries to carry out a calculation too quickly.

For this reason it makes good sense to have a method of estimating the size of an answer as a check on the one produced by the calculator.

One way of doing this is to round off the numbers involved. Under normal circumstances it is sufficient to round off a number between 10 and 100 to the nearest ten or a number between 100 and 1000 to the nearest hundred and so on.

Example 1

If we know that the number of matches in a box is advertised as 70 to the nearest 10 then:

- the smallest number to be ***rounded up*** to 70 is 65
- the largest number to be ***rounded down*** to 70 is 74 (75 would be rounded ***up*** to 80)

Therefore, in reality there could be between 65 and 74 (inclusive) matches in the box.

Example 2

If the number of people in a village in India is 800 to the nearest 100 then:

- the smallest number to be ***rounded up*** to 800 is 750
- the largest number to be ***rounded down*** to 800 is 849 (850 would be rounded ***up*** to 900)

Therefore, in reality there could be between 750 and 849 (inclusive) people in the village.

Example 3

Consider the number of spectators in a football ground. If we know that the number present is 45000 to the nearest 1000 then:

- the smallest number to be ***rounded up*** to 45000 is 44500

$44452 \cong 44000$ rounded to the nearest 1000
$44452 \cong 44450$ rounded to the nearest 10
$44452 \cong 44500$ rounded to the nearest 1000

45499 is rounded to the nearest 10 as 45500
45499 is rouded to the nearest 100 as 45500
45 499 is " " " " 1000 as 45000

- the largest number to be ***rounded down*** to 45000 is 45499 (45500 would be rounded ***up*** to 46000)

Therefore, in reality there could be between 44500 and 45499 (inclusive) people in the ground.

Rounding Off to a Certain Number of Decimal Places

It is the number of figures ***after*** the decimal point that describes the number of decimal places.

32.6 is a number that has 1 decimal place (abbreviation: dp)

41.213 is a number that has 3 decimal places.

To correct a number to so many decimal places, if the first figure to be left out is 5 or more then the previous figure is increased by 1.

Example 1 Correct 15.687 to 2 decimal places

Here the figure to be left out is 7 and since this is ***above*** 5 the previous figure 8 is increased by 1 to 9. So the answer becomes 15.69 (2dp).

If the first figure to be left out is ***below*** 5 then the previous figure is left unaltered.

15.687 to round it to 2 dp 15.69
15.684 " " " " " " 15.68
15.687 to " " " 1 " 15.7

Example 2 Correct 36.124 to 2 decimal places

Here the first figure to be left out is 4 and since this is below 5 the previous figure is left unaltered. So the answer becomes 36.12 (2 dp)

Correct 36.124 to 1 decimal place
36.1

Example 3

Correct 0.0657 to 2 decimal places	The answer is 0.**07** (2dp)
Correct 17.401 to 2 decimal places	The answer is 17.**40** (2dp)

Further Examples

Correct the following to 3 decimal places:

Number	Answer
16.8965	16.897
4.8995	4.900
3.9999	4.000
0.0765	0.077
0.0058	0.006
0.0019	0.002

Notice how the zeros are kept to show that they are one of the decimal places.

The zeros are kept to show the position of the decimal point

Rounding to a Certain Number of Significant Figures

In any number the most significant figure is the one that has the greatest value. In most cases this is the leftmost one. The exception to this is when this is a zero.

Consider the number 1589. The most significant figure is the 1, because it represents 1000. The 9 only represent 9 units.

In the number 0.006218, the most significant figure is the 6. The zeros at the beginning of a decimal number are not counted as significant figures but they must be included in the final answer.

The rules for correcting numbers to so many significant figures (***abbreviation* s.f.**) are very similar to those for decimal places.

If the first figure to be left out is 5 or more then the previous figure is increased by 1. If below 5 the previous figure is left unaltered.

Example 1

Correct 23.685 to 3 significant figures	The answer is 23.7 (3 s.f.)
Correct 14.102 to 3 significant figures	The answer is 14.1 (3 s.f.)

As stated zeros must be kept to show the position of the decimal point.

0.00568	corrected to 2 significant figures would be	0.**00**57
0.01237	corrected to 2 significant figures would be	0.**0**12

Zeros must also be kept to show that they are one of the significant figures!

Example 2

37682 corrected to 2 significant figures would be 38**000**

The reason for this is quite obvious. If you had £37682 in the bank, it would be ludicrous to say you had £38 corrected to 2 significant figures.

37682 corrected to 3 signifficant figures would be 37700

Further Examples

Correct the following to 3 significant figures.

Number	Answer
21.069	21.1
0.006938	0.00694
0.009999	0.0100= 0.010 = 0.01
39,990	40,000
56843	56800

Estimations

A calculator in good working condition should be completely reliable. The user however is prone to error. It is quite easy to miss a digit or get a decimal point in the wrong place if working at speed. It makes sense therefore to estimate the answer. When estimating, numbers should be chosen that add, subtract, multiply or divide easily.

Example 1 51.9 × 2.79

A rough estimate would be 50 × 3 = 150.

The correct answer is 144.80 (2dp).

The rough estimate is reasonable and shows us that the correct answer is

144.80 and not 14.480 or 1448.0.

Example 2 Subtract 17.86 from 59.65.

A rough estimate would be 60 – 18 = 42. The correct answer is 41.79

Once again the rough estimate tells us the answer is sensible.

Example 3

Evaluate $\frac{62.5 \times 29.3}{14.7 \times 9.6}$

A rough estimate would be: $\frac{60 \times 30}{15 \times 10}$

which can be cancelled down quite easily to: $\frac{\overset{6}{\cancel{60}} \times \overset{2}{\cancel{30}}}{\underset{1}{\cancel{15}} \times \underset{1}{\cancel{10}}} = \frac{12}{1} = 12$

The correct answer is 12.98 (2dp).

The answer is sensible and tells us we have the figures correct and the decimal point is in the correct place.

Example 4

$$\frac{39.65 \times 1.87}{0.98}$$

$$\frac{40 \times 2}{1} = 80$$

The correct answer is 75.66 (d.p.). Again, the answer is sensible.

Examples:

$$\frac{32.44 \times 19.75 \times 21.15}{2.25 \times 1.91 \times 11.92} \simeq \frac{30 \times \overset{10}{\cancel{20}} \times \cancel{20}}{\cancel{2} \times \underbrace{2 \times 10}_{\cancel{20}}} \simeq 300$$

$$= 264.5247$$

Standard Form

Before we look at this topic we need to have a firm understanding of what ***'powers of ten'*** are. As we have seen previously; 2 to the power 4, written $2^4 = 2 \times 2 \times 2 \times 2 = 16$.

In a similar fashion:

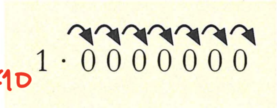
10 to the power 1, written $10^1 = 10$

10 to the power 2, written $10^2 = 10 \times 10 = 100$

10 to the power 3, written $10^3 = 10 \times 10 \times 10 = 1000$

If you start with 1.0 you can see quite clearly that the power tells you the number of places to move the decimal point to the ***right***. Each time we move one place we have ***multiplied*** by 10.

So 10^7 would be



which gives 10000000

10x10x10x10x10x10x10

10000000

Get the idea? If you don't, then read the previous paragraph again. It's not that difficult really.

In a similar way if we divide a number by 10 it makes sense that we move the decimal point one place to the ***left***.

So: 10 to the power minus 1, written $10^{-1} = 0.1$

0.01

10 to the power minus 2, written $10^{-2} = 0.1 \times 0.1 = 0.01$

10 to the power minus 3, written $10^{-3} = 0.1 \times 0.1 \times 0.1 = 0.001$

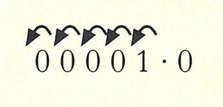
and so on.

$10^{-5} = 0.\underline{0}\,\underline{0}\,\underline{0}\,\underline{0}\,\underline{1}$

$10^5 = 1\,\underline{0}\,\underline{0}\,\underline{0}\,\underline{0}\,\underline{0}$

Again, starting with 1.0 you can see that the power tells you the number of places to move the decimal point to the ***left***. Each time we move one place we have ***divided*** by 10.

So 10^{-5} would be



which gives 0.00001

Definition of Standard Form

Any large number (e.g. 9800000000) or small number (e.g. 0.000000000089) can be written as a number between 1 and 10 multiplied by a power of 10.

When doing arithmetical operations with such numbers, the repeating of zeros could lead to errors because it would be very easy to leave a nought out or put an extra one in. Expressing such numbers in ***Standard Form*** or ***Scientific Notation*** as it is sometimes called can diminish the possibility of such errors.

$9800000000 = 9.8 \times 10^9 = 98 \times 10^8 \Rightarrow$ in scientific calculator this is 9.8+E09

Example 1

The velocity of light is 280,000,000 metres per second. Write this number in Standard Form Notation.

2.8×10^8

Step 1

First of all write down the number as a number between 1 and 10. This involves moving the decimal point. So 280,000,000.0 becomes 2.8.

Clearly these two numbers are not equal so:

Step 2

Count the number of places the decimal point has to be moved to the ***right*** for 2.8 to become 280,000,000.0 remembering that each time you move to the right you are ***multiplying by 10***. The answer is 8 places.

Step 3

The number of places moved becomes the power of 10.

So 280,000,000.0 is standard form notation is 2.8×10^8

Example: $388\,000\,000\,000\,00 = 3.88 \times 10^{13}$

Example 2

Write 0.0000000078 in Standard Form.

Step 1

As before write down this number as a number between 1 and 10. So, 0.0000000078 becomes 7.8.

But once again these numbers are not equal so:

Step 2

Count the number of decimal places the decimal point has to be moved to the ***left*** for 7.8 to become 0.0000000078, remembering that this time each time you move one place to the left you are ***dividing by 10***. The answer is 9 places.

Step 3

Once again the number of places moved becomes the power of 10 but since we have moved to the left this time we put a minus (-) sign in front of the power. So, $0.0000000078 = 7.8 \times 10^{-9}$

Example: $0.000006349 = 6.349 \times 10^{-6}$

Conversion of Numbers in Standard Form to Normal Numbers

Example 1 Convert 1.98×10^5 into a normal number.

1.98×100000
198000

The power to which the 10 is raised (called the ***index*** incidentally) is 5,

so the ***decimal point*** has to be moved 5 places to the ***right***..

$1{\cdot}98 \times 10^5$

1 · 9 8 0 0 0

i.e. 198000

Example 2 Convert 3.26×10^{-4} into a normal number.

0.000326

This time the power to which the 10 is raised (***the index***) is -4, so the decimal point has to be moved 4 places to the ***left***.

$3{\cdot}26 \times 10^{-4}$

0 0 0 3 · 2 6

i.e. 0.000326

NB. Always put a zero in front of the decimal point in these cases.

Addition and Subtraction of Numbers in Standard Form

To achieve this without the aid of a calculator the standard form numbers should be converted into 'normal' numbers, added/subtracted and then converted back into standard form.

Example 1

Evaluate $1.25 \times 10^3 + 3.27 \times 10^4$, giving your answer in standard form.

Now $1.25 \times 10^3 = 1250$ $\Rightarrow 1.25 \times 1000 = 1250$

and $3.27 \times 10^4 = 32700$ $\Rightarrow 3.27 \times 10000 = 32700$

So $1.25 \times 10^3 + 3.27 \times 10^4$

$= 1250 + 32700$

$= 33950$

$= 3.3950 \times 10^4$

$1{\cdot}25 \times 10^3 + 3{\cdot}27 \times 10^4$

$$\begin{array}{ccccc} \downarrow & & \downarrow & & \\ 1250 & + & 32700 & = & 33950 \\ & & & & \downarrow \end{array}$$

As standard form: $3{\cdot}395 \times 10^4$

$$\begin{array}{r} 32700 \\ +1250 \\ \hline 33950 \end{array}$$

$3.3950 \times 10^4 = 3.3950000 \times 10^4$
$= 3.395 \times 10^4$