Example 1

The binary number : 1011.011 represents

$$\begin{array}{cccc} 2^3 & 2^2 & 2^1 & 2^0 \\ \hline 1 & 0 & 1 & 1 \end{array} \cdot \begin{array}{ccc} 2^{-1} & 2^{-2} & 2^{-3} \\ \hline 0 & 1 & 1 \end{array}$$

$=1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-3}$

$= (1 \times 8) + (0 \times 4) + (1 \times 2) + (1 \times 1) + (0 \times \frac{1}{2}) + (1 \times \frac{1}{4}) + (1 \times \frac{1}{8})$

$= 8 + 0 + 2 + 1 + 0 + 0.25 + 0.125$

$=11.375$

Example 2

111.01101 can be represented as:

$1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-3} + 0 \times 2^{-4} + 1 \times 2^{-5}$

$= (1 \times 4) + (1 \times 2) + (1 \times 1) + (0 \times \frac{1}{2}) + (1 \times \frac{1}{4}) + (1 \times \frac{1}{8}) + (0 \times \frac{1}{16}) + (1 \times \frac{1}{32})$

$= 4 + 2 + 1 + 0 + 0.25 + 0.125 + 0 + 0.03125$

$= 7.40625$

Arithmetical Operations with Binary Numbers

Addition

Remember from our work in the binary section, the largest value we can have in any column of a binary calculation is 1.

The columns are:

16	8	4	2	1
2^4	2^3	2^2	2^1	2^0

a) $1 + 0 = 1_2$ (no problem here, sum does not exceed 1).

b) $1 + 1 = 10_2$

In the 2^0 column $1 + 1 = 10$, put down 0

Carry 1 to the 2^1 column

2^1	2^0
	1
	1
1	0

c) $1 + 1 + 0 = 10_2$

In the 2^0 column $1 + 1 + 0 = 10$. Put down 0

Carry 1 to the 2^1 column $1 + 0 = 1$

Put down 1

2^1	2^0
	1
	1
	0
1	0

d) $1 + 1 + 1 + 1 = 100_2$

In 2^0 column $1 + 1 + 1 + 1 = 2^2$, so put 0 in the 2^0 column, 0 in the 2^1 column and 1 in the 2^2 column.

2^2	2^1	2^0
		1
		1
		1
		1
1	0	0

Example 1 $1 + 11 = 100_2$

In 2^0 column $1 + 1 = 10$, put down 0 carry 1

In 2^1 column $1 + 1$ (carried) $= 10$, put down 0, carry 1

In 2^2 column $0 + 1$ (carried) $= 1$

2^2	2^1	2^0
0	0	1
0	1	1
1	0	0

Example 2 $111 + 10 = 1001_2$

In 2^0 column $1 + 0 = 1$, put down 1.

In 2^1 column $1 + 1 = 10$, put down 0 carry 1.

In 2^2 column $1 + 1$(carried) $= 10$, put down 0, carry 1.

In 2^3 column $0 + 1$(carried) $= 1$

2^3	2^2	2^1	2^0
	1	1	1
		1	0
1	0	0	1

Example:

$$\begin{array}{r} {}^{1\,1\,1\,1\,1\,1\,1} \\ 101110011 \\ +\ 110111111 \\ \hline 110010010 \end{array}$$

Subtraction

When subtracting in binary, instead of borrowing tens, we borrow twos.

Example 1 $1 - 0 = 1_2$ (no sweat!)

Example 2 $11 - 1 = 10_2$

In 2^0 column $1 - 1 = 0_2$

In 2^1 column $1 - 0 = 1_2$

2^1	2^0
1	1
0	1
1	0

Example 3 $10 - 1 = 01_2$

In 2^0 column 0 - 1, cannot do, so borrow 2.

From next column (2 - 1 =1)

In 2^1 column 0 - 0 =0

2^1	2^0
$\cancel{1}$	${}^{2}0$
0	1
0	1

Example 4 1010 - 111 = 0011

In 2^0 column 0 - 1 cannot do, so borrow 2 from next column (2 - 1 =1).

In 2^1 column 0 - 1 cannot do, so borrow 2 from 2^2 column.

There is a 0 here so borrow 2 from 2^3 column.

This puts 2×2's into the 2^2 column, from which you can take one of them putting 2 into the 2^1 column.

In 2^2 column 1 - 1 = 0. In 2^3 column 0 - 0 = 0.

2^3	2^2	2^1	2^0
$\cancel{1}$	${}^{1}\cancel{{}^{2}}0$	${}^{2}\cancel{1}$	${}^{2}0$
	1	1	1
0	0	1	1

Example:

```
  10110011
- 00111101
----------
  01010110
```


Multiplying Binary Numbers

You need to remember three things here: $1 \times 1 = 1$ $1 \times 0 = 0$ $0 \times 1 = 0$

Example 1 100101 × 11

Multiply by the right-hand 1 and put down answer (a).

Put down a 0 and multiply by 1, put down answer (b).

Finally, add the two lines together (a + b)

		1	1	1	1	0	1	
					x	1	1	
		1	1	1	1	0	1	(a)
	1	1	1	1	0	1	0	(b)
1	0	1	1	0	1	1	1	

This is very similar to denary multiplication.

Example 2 110101 × 10

Multiplying by right hand 0 gives a row of zeros. (a)

Put down a 0 and multiply by 1 (b).

Finally add (a) + (b) lines.

	1	1	0	1	0	1	
				x	1	0	
	0	0	0	0	0	0	(a)
1	1	0	1	0	1	0	(b)
1	1	0	1	0	1	0	

Division of Binary Numbers

This is very similar to long division in denary.

Example 1 110 ÷ 11

11 into 1 will not go. 11 into 11 goes once.

Put a 1 over right-hand 1. Subtract 11 - 11 =00.

Bring down next digit 0. 11 into 000 will not go.

Put down a 0 in answer.

```
      10
 11)110
    11
    000
    000
    ...
```

Example 2 1110111 ÷ 111

111 into 1 will not go. 111 into 11 will not go.

111 into 111 goes once.

Put a 1 over third digit from left.

Subtract 111 - 111=000.

Bring down the 0. 111 into 0 will not go.

Put a 0 on answer line.

Bring down next 1.

111 into 1 will not go, put a 0 on the answer line.

```
       10001
111 )1110111
     111↓
     0000111
         111
     0000000
```

Bring down next 1. 111 into 11 will not go, put a 0 on the answer line

Bring down next 1. 111 into 111 goes once, put a 1 on the answer line next to the 0 and put 111 under the 111 to give 000.

(111 - 111 = 000)

The division is complete.

Arithmetical Operations with Octal and Hexadecimal Systems

Octal

Since with octal we are working in base 8, the highest number we can have in any column is 7 (i.e. It must always be one less than the base number).

The octal columns are:

4096	512	64	8	1
8^4	8^3	8^2	8^1	8^0

Addition of Octal Numbers

Example 1

In 8^0 column 4 + 3 = 7. No problem here, the answer does not exceed 7, so put down 7.

$$\begin{array}{cc} 8^1 & 8^0 \\ & 4 \\ & 3 \\ \hline & 7 \end{array}$$

Example 2

In 8^0 column $7 + 4 = 11_{10} = 1 \times 8 + 3$ rem.

We put the 3 in the 8^0 column and carry the 1 to the 8^1 column.

There 0 + 1 (carried) =1.

Put the 1 down.

8^1	8^0
	4
	7
1_1	3

Example 3

In 8^0 column $7 + 2 = 9_{10} = 1 \times 8 + 1$ rem.

Put down the 1 in the 8^0 column and carry the 1 to the 8^1 column.

Here 3 + 4 + (1) carried = $8_{10} = 1 \times 8 + 0$ rem.

Put down the 0 in the 8^1 column and carry the 1 to 8^2 column.

Here 0 + 1 (carried) = 1. Put down the one.

8^2	8^1	8^0
	3	7
	4	2
1	0	1
1	1	

Example:

1 5 8 6 6 7

$$\begin{array}{r} 475\cancel{6}7 \\ +36573 \\ \hline 106362 \end{array}$$

10 − 8 = 2
14 − 8 = 6
11 − 8 = 3
14 − 8 = 6
8 − 8 = 0

Example 4

In 8^0 column $6 + 7 = 13_{10} = 1 \times 8$ rem.5.

Put 5 down and carry 1to 8^1 column.

Here 5 + 4 + (1) carried $= 10_{10} = 1 \times 8$ rem. 2.

Put 2 down in 8^1 column and carry 1 to 8^2 column.

Here 4 + 3 + 1(carried) $= 8_{10} = 1 \times 8$ rem. 0.

Put 0 down in 8^2 column and carry 1 to 8^3 column.

Here 0 + 1(carried) =1. Put 1 down in 8^3 column.

8^3	8^2	8^1	8^0
	4	5	6
	3	4	7
1	0	2	5
	1	1	

Subtraction of Octal Numbers

When we did addition of octal numbers, groups of 8 had to be carried. When we do subtraction groups of 8 have to be borrowed.

Example 1

In 8^0 column 2 - 5 cannot be done. Borrow one group of 8 from 8^1 column (leaving 2 in the 8^1 column).

We then add the 'borrowed' 8 to the 2 in the 8^0 column making 10, thus 10 - 5 = 5.

Put the 5 in the 8^0 column.

In the 8^1 column then 2 - 2 = 0. Put 0 down in 8^1 column.

In 8^2 column 4 - 3 = 1. Put in 8^2 column.

Finally in 8^3 column 5 - 1 = 4. Put 4 down in 8^3 column to complete the answer.

8^3	8^2	8^1	8^0
5	4	$^2\cancel{3}$	2
1	3	2	5
4	1	0	5

Example 2

In 8^0 column 2 - 7 cannot be done.

Borrow one group of 8 from 8^1 column (leaving 2 in the 8^1 column).

We then add the borrowed 8 to the 2 in the 8^0 column

making 10, then 10 - 7 = 3. Put 3 down in the 8^0 column.

In the 8^1 column 2 - 6 cannot be done.

Borrow one group of 8 from 8^2 column (leaving 4 in 8^2 column)

We then add the borrowed 8 to the 2 in the 8^1 column making 10, then 10 - 6 = 4. Put the 4 down in the 8^1 column.

Finally in the 8^2 column 4 - 3 = 1. Put down 1 to complete the answer.

8^2	8^1	8^0
4 $\cancel{5}$	2 $\cancel{3}$	2
3	6	7
1	4	3

8+2-7=3

Example:

	5	2		6	4	
	$\cancel{6}$	$\cancel{3}$	0	$\cancel{7}$	$\cancel{5}$	2
-	5	7	3	6	6	4
	0	3	5	0	6	6

Hexadecimal

Since with hexadecimal we are working in base 16, the highest number we can have in any column is 15 (remember it must be one less than the base number). Now remember in base 16, 15 is F. Before we start adding and subtracting hexadecimal numbers it is worth reminding ourselves of the 15 digits used.

They are: 1, 2, 3, 4, 5, 6, 7, 8, 9. A, B, C. D, E, F
10 11 12 13 14 15

The hexadecimal columns are:

4096	256	16	1
16^3	16^2	16^1	16^0

Addition of Hexadecimal Numbers

1, 2, 3, 4, 5, 6, 7, 8, 9. A, B, C. D, E, F
10 11 12 13 14 15

Example 1

In the 16^0 column $5 + 8 = 13_{10}$ (i.e. D in Hex).

D is put down in the 16^0 column.

In the 16^1 column $3 + 4 = 7$.

Put down the 7 in 16^1 column to complete the answer.

16^1	16^0
3	5
4	8
7	D

1, 2, 3, 4, 5, 6, 7, 8, 9. A, B, C. D, E, F
10 11 12 13 14 15

Example 2

In 16^0 column $(C_{16}=12) + 2 = 14(14 = E_{16})$.

Put E down in the 16^0 column.

In the 16^1column $6 + 4 = 10$ $(10 = A_{16})$.

Put A in the 16^1 column to complete the answer.

16^1	16^0
4	C
6	2
A	E

Example 3

In 16^0 column $(C_{16}=12 + F_{16}=15) = 27_{10}$.

Now $27_{10} = 1 \times 16$ rem. 11, so put down 11 (B_{16}) in the 16^0 column and carry 1.

1, 2, 3, 4, 5, 6, 7, 8, 9. A, B, C. D, E, F
10 11 12 13 14 15

In the 16^1 column $8 + (15 = F_{16}) + 1(\text{carried}) = 24$.

Now $24_{10} = 1 \times 16$ rem. 8, so put down 8 in 16^1 column and carry 1.

In 16^2 column $1 + 3 + 1(\text{carried}) = 5$.

Put down 5 in 16^2 column to complete answer

16^2	16^1	16^0
1	8	C
3	F	F
5	8	B
1	1	

Example:

$$\begin{array}{r} 1\;8\;\;\;4\;\;6\;\; \\ F\;\cancel{7}\;6\;\cancel{3}\;A\;\cancel{5}\;D \\ +\;\;1\;9\;A\;B\;D\;9\;6 \\ \hline 1\;1\;1\;0\;F\;7\;F\;3 \end{array}$$

Subtraction of Hexadecimal Numbers

Example 1

1, 2, 3, 4, 5, 6, 7, 8, 9. A, B, C. D, E, F
10 11 12 13 14 15

In the 16^0 column ($B_{16} = 11$) - 9 = 2.

Put down the 2 in the 16^0 column.

In the 16^1 column 3 - 1 = 2.

Put 2 down in the 16^1 column to complete the answer.

16^1	16^0
3	B
1	9
2	2

Example 2

1, 2, 3, 4, 5, 6, 7, 8, 9. A, B, C. D, E, F
10 11 12 13 14 15

In the 16^0 column 3 - 8 cannot be done.

Borrow one group of 16 from 16^1 column (leaving 4 in the 16^1 column).

We then add the borrowed 16 to the 3 in the 16^0 column making 19, then 19 - 8 = 11 (B_{16}). Put B down in 16^0 column.

In the 16^1column 4 - 2 = 2. Put 2 down in 16^1 column to complete the answer.

16^1	16^0
$^{4}\cancel{5}$	3
2	8
2	B

Example 3

In 16^0 column ($F_{16} = 15$) - 8 = 7.

Put down 7 in 16^0 column.

In 16^1 column ($E_{16} = 14$) - ($B_{16} = 11$) = 3.

Put down 3 in 16^1 column to complete the answer.

16^1	16^0
E	F
B	8
3	7

1, 2, 3, 4, 5, 6, 7, 8, 9. A, B, C. D, E, F
10 11 12 13 14 15

Example 4

16^2	16^1	16^0
$^{D}\cancel{E}$	$^{B}\cancel{C}$	A
2	F	F
B	C	B
11	12	11

6 22 E 20 F 17
A $\cancel{7}$ $\cancel{6}$ D $\cancel{F}$ $\cancel{5}$ $\cancel{0}$ $\cancel{1}$
- 6 5 F A 0 9 F F
4 1 7 3 E B 0 2

In the 16^0 column ($A_{16} = 10$) - ($F_{16} = 15$) cannot be done, so borrow one group of 16 from the 16^1 column (leaving $B_{16} = 11$) in the 16^1 column.

We then add the borrowed 16 to the A in the 16^0 column making $16 + (A_{16} = 10) = 26$. Then $26 - (F_{16} = 15) = 11$. Now $(11 = B_{16})$.

Put this down in 16^0 column. In 16^1 column ($B_{16} = 11$) - ($F_{16} = 15$) cannot be done, so borrow one group of 16 from 16^2 column (leaving ($E_{16} = 14$) $- 1 = 13$(D)).

We then add the borrowed 16 to the B in the 16^1 column making $16 + (B_{16} = 11) = 27$.

$(372)_8$
$\times\ (66)_8$
$- - \ .\ 4$

Then $27 - (F_{16} = 15) = 12$ which is C_{16}.

Put C_{16} down in 16^1 column.

Finally, in the 16^2 column ($D_{16} = 13$) $-2 = 11$ (B_{16}).

Put B_{16} down in the 16^2 column.

Binary Coded Decimal (BCD)

As the name suggests BCD uses a binary code to represent the decimal digits. It is a 4-bit code and is only used for the representation of numeric values.

Each of the ten digits used in the decimal system is coded with its binary equivalent as follows:-

Dec. Digit	0	1	2	3	4	5	. . .	9
BCD Code	0000	0001	0010	0011	0100	0101	. . .	1001

Any number can be represented by coding each digit separately.

Example 1

A decimal value of 624 would be coded as follows:

Decimal	6	2	4
BCD Code	0110	0010	0100

There are ***two points*** that need to be made regarding this particular coding method:

- Only ten of the possible sixteen combinations of 4-bits are used.
- BCD uses more bits generally speaking and thus more storage than a pure binary representation of the number.

Example 2

Consider the decimal number 1265_{10}.

In BCD it is:- $\underbrace{0001 \quad 0010 \quad 0110 \quad 0101}_{16 \text{ bits}}$

In pure Binary it is:-

1024	512	256	128	64	32	16	8	4	2	1
1	0	0	1	1	1	1	0	0	0	1

11 bits

It is therefore more economical to store in pure binary. There is a fairly simple relationship between the code for a positive number and the corresponding negative number. It is called ***twos complement***.

Twos Complement

In two complement coding, the bits have the same place values as binary numbers except that the most significant bit (the leftmost bit) represents a negative quantity.

The place values for a six bit, twos complement numbers are:

-32	16	8	4	2	1
1	0	0	1	1	0

This equals -32 + 4 + 2 = -26

So if we consider +11 in 6 bits we would have: $0\ 01011_2$. The corresponding negative number -11 would be 110101.

	-32	16	8	4	2	1	
+11	0	0	1	0	1	1	
-11	1	1	0	1	0	1	(-32+21)

It can be seen that except for the units column the bits for the negative number are the opposite of those for the positive number. So to change from a positive number to a negative number in twos complement form:

Change the 0's to 1's and the 1's to 0's and then add 1.

	32	16	8	4	2	1	
+8	0	0	1	0	0	0	
-8	1	1	0	1	1	1	(bits changed)
						1	(add)
	1	1	1	0	0	0	

Subtraction Using Twos Complement

Since subtraction can be considered as the addition of a negative number (16 - 7 ≡ 16 +(-7). we can use twos complement to do subtraction by making the ***second number negative*** as described previously and then ***adding this to the first number.***

Example 25 - 16 using twos complement.

	-32	16	8	4	2	1	
16	0	1	0	0	0	0	
Reverse bits	1	0	1	1	1	1	
Add 1						1	
-16	1	1	0	0	0	0	
Add 25	0	1	1	0	0	1	
	0	0	1	0	0	1	= 9

(1) This extra bit carried over from the -32 column is called the overflow bit and in this case is ignored

Ones Complement

The twos complement of a binary number is, as we have seen, found by reversing the bits and adding 1.

Ones complement is found however by merely reversing the bits.

Example

	-31	16	8	4	2	1	
+7	0	0	0	1	1	1	
-7	1	1	1	0	0	0	(-31+24=-7)

Note: In this type of coding, the most significant bit, the leftmost bit, now represents -31 rather than -32 as in twos complement.

Subtraction Using Ones Complement

This is similar to subtraction using twos complement except that in this case, the overflow bit must be added back to the units column.

Example 25 - 16 using ones complement.

	-31	16	8	4	2	1	
16	0	1	0	0	0	0	
-16	1	0	1	1	1	1	
add 25	0	1	1	0	0	1	
	0	0	1	0	0	0	
						1	
			1	0	0	1	= 9

overflow
(1)

Simultaneous Linear Equations

If we consider the two linear equations: $3x - 2y = 4$ and $2x + 3y = 7$

Then each equation contains x and y which are unknown quantities.