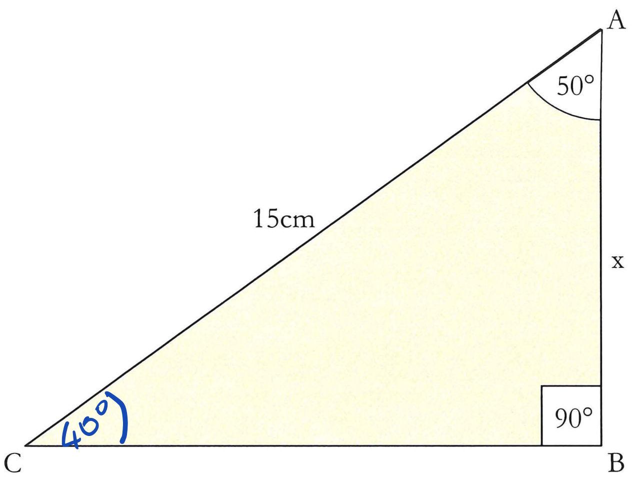
Example

Find AB



$\frac{AB}{15} = \cos 50°$

$AB = 15 * \cos 50° = 9.64$

$\frac{BC}{15} = \sin 50°$

$BC = 15 \sin 50° = 11.49$

Clearly AB is the required side. AC is the given side so BC is ***not*** required. In this problem BC is the opposite side: -

$\sin 40° = \frac{x}{15}$

Cross out ratios with opposite:

$x = 15 \sin 40°$

$x = 9.64$

Sin = ~~Opposite~~ / ~~Hypotenuse~~   Cos = Adjacent / Hypotenuse   Tan = ~~Opposite~~ / ~~Adjacent~~

Cos is the required ratio

$\text{Cos } 50° = \frac{x}{15}$

$x = \text{Cos } 50 \times 15$

$x = 9.6\text{cm (1 dp)}$

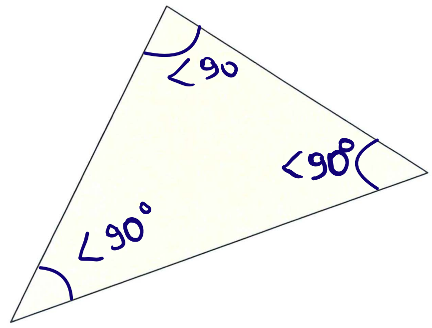
Types of Triangle

A polygon is a plane figure bounded by straight lines. Therefore a triangle (3 sides) is the smallest polygon possible. The sum of the angles of a triangle is 180^0 irrespective of its shape or size.

In any triangle, the longest side always lies opposite to the largest angle and the shortest side always lies opposite to the smallest angle. It follows that if two sides of a triangle are equal, then the angles opposite those sides must also be equal.

a) Acute Angled Triangle

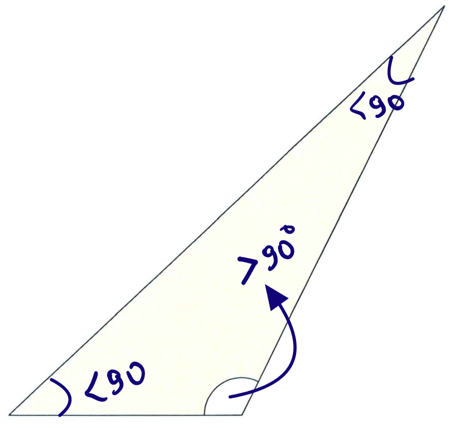
As the name suggests an acute angled triangle has each of its three angles less than 90°.



b) Obtuse Angled Triangle

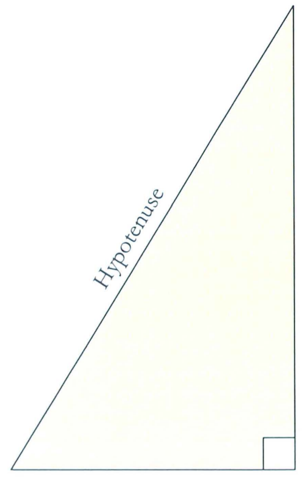
This has one of its angles greater than 90°.

NB: It is impossible to have two angles greater than 90°, since the three angle sum is always 180°.



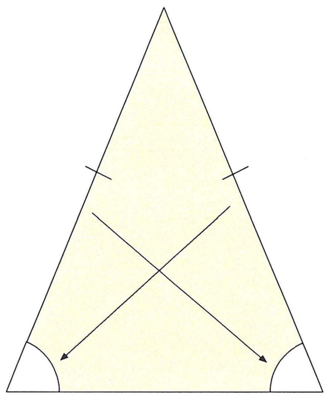
c) Right Angled Triangle

In this type of triangle, one angle is equal to 90°. The side opposite to it is called the hypotenuse and is the longest side in the triangle.



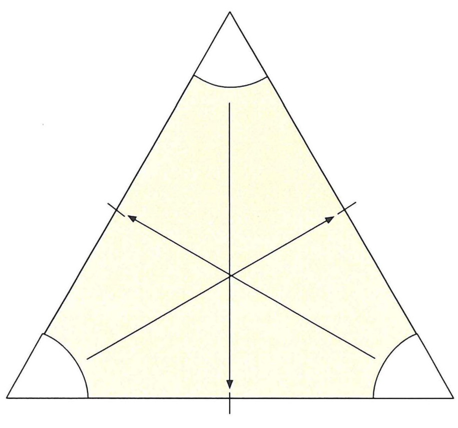
d) Isosceles Triangle

This type of triangle has two equal sides and therefore the angles opposite these sides must also be equal.



e) Equilateral Triangle

This type of triangle has three equal sides and therefore the angles opposite these sides must also be equal. Since the angle sum is 180°, then each of these angles must be 60°.



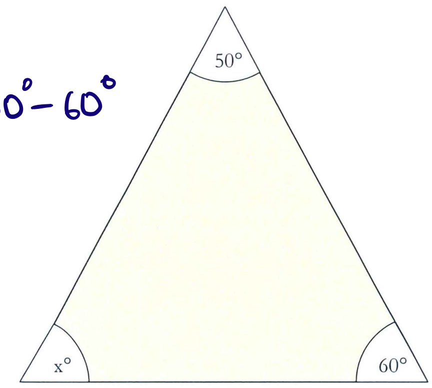
f) Scalene Triangle

This is a more general term for a triangle that has three sides of different length. It follows therefore that the three angles must be different sizes. Only acute angled, obtuse angled and some right-angled triangles are in this category.

Example 1

Find the angle marked x in the following triangle:

$50° + 60° + x = 180°$
$x = 180° - 50° - 60°$
$x = 70°$



Since the three angles of a triangle add up to 180° then:-

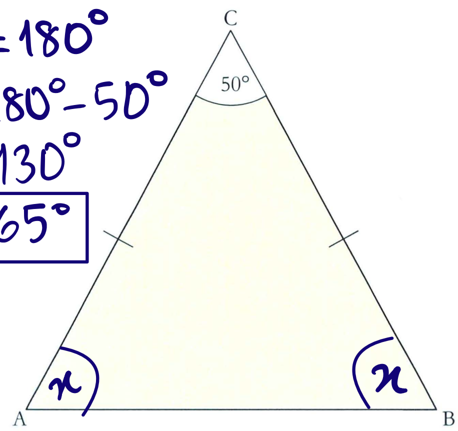
$x = 180° - 50° - 60°$

$x = 70°$

Example 2

In the triangle ABC, AC = BC and C = 50°. Find the angles A and B

$x + x + 50° = 180°$
$2x = 180° - 50°$
$2x = 130°$
$x = 65°$



Since AC=BC, the triangle is isosceles and therefore the angles opposite the equal sides ∠A and ∠B must be equal.

So ∠A + ∠B = 180° - 50° = 130°

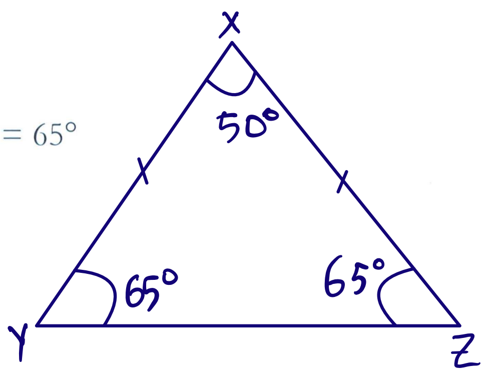
and $\angle A = \angle B = \frac{130}{2} = 65°$

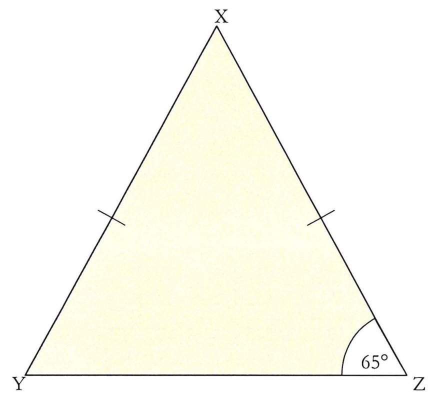
Example 3

In the triangle XYZ, XY = XZ and ∠XZY = 65°

a) Find ∠XYZ = 65°

b) Find ∠YXZ = 180 - 65 - 65
= 50





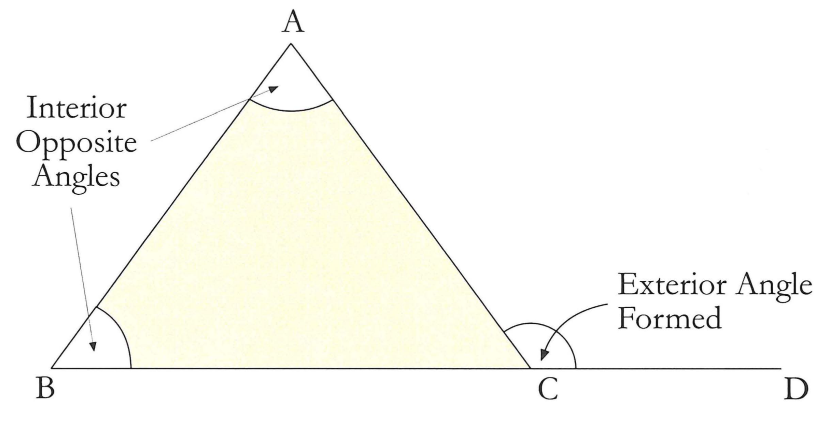
Since XY = XZ, the triangle is isosceles and the angles opposite these equal sides ∠XYZ and ∠XZY must also be equal.

So since ∠XZY = 65°, then ∠XYZ = 65° also.

The three angles in the triangle add up to 180°

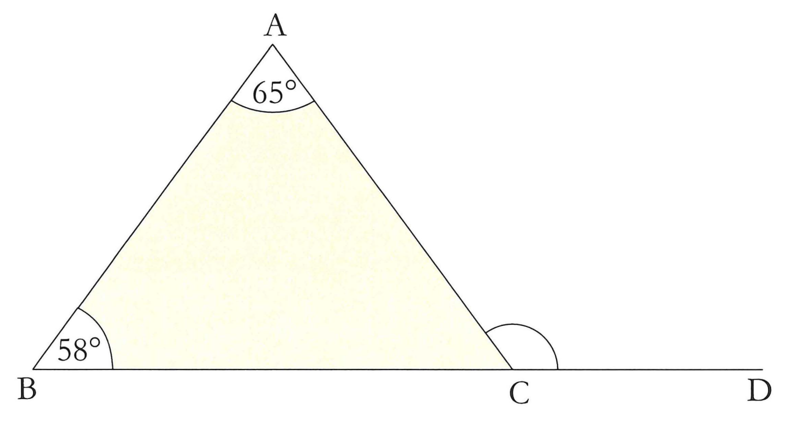
∴ ∠YXZ = 180° - 65° - 65° = 50°

When one side of a triangle is extended, then the external angle so produced is equal to the sum of the two interior opposite angles.



In other words, ∠ACD = ∠ABC + ∠BAC

Consider the following:-



Since the angle sum of a triangle is 180°, then $\angle ACB = 180° - 58° - 65° = 57°$

Now BCD is a straight line, so $\angle ACB + \angle ACD = 180°$

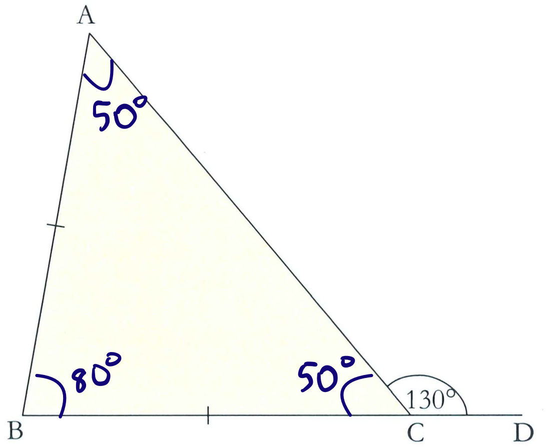
Since $\angle ACB = 57°$, then $\angle ACD = 180° – 57° = 123°$

$\angle ACD$ (external angle)$=\angle ABC + \angle BAC$

$$123° = 58° + 65°$$

Example 4

In the following diagram, find the angle $\angle ABC$



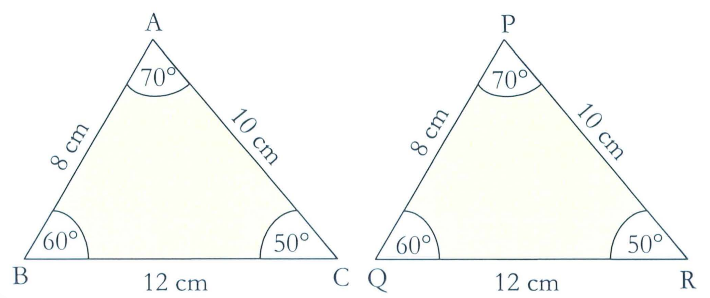
$\angle ACB = 180° - 130° = 50°$ (BCD is a straight line $\therefore$ $\angle ACB + \angle ACD = 180°$)

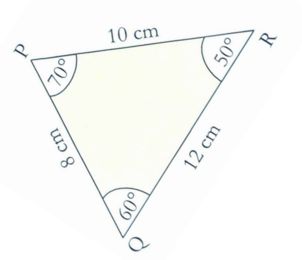
Now since AB = BC the triangle ABC is isosceles and $\angle BAC = \angle ACB = 50°$

$\angle ABC = 180° - 50° - 50° = 80°$

Congruent Triangles

Consider the two triangle ABC and PQR:





We can see that:	AB=PQ	also	$\angle A = \angle P$
	BC=QR	also	$\angle B = \angle Q$
	AC=PR	also	$\angle C = \angle R$

The six elements (the three sides and the three angles) of ΔABC are respectively equal to the six element of ΔPQR. Such triangles are said to be congruent and we write: $\Delta ABC \equiv \Delta PQR$ ($\equiv$ meaning congruent)

To prove that two triangles are congruent it is not necessary to prove all six equalities. Any one of the following four conditions is sufficient: -

- Three sides in one of the triangles equal to three sides in the other.
- Two sides and the included angle (the angle between them) in one of the triangles equal to two sides and the included angle in the other.
- One side and two angles in one of the triangles equal to one side and two similarly placed angles in the other.
- In the case of two right-angled triangles the hypotenuse and one other side in one of the triangles equal to the hypotenuse and one other side in the other.

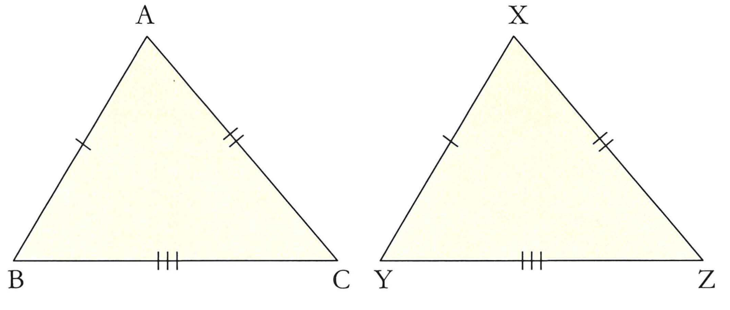
It should be noted that there are three conditions that ***do not*** prove that two triangles are congruent.

- The three angles in one of the triangles equal to the three angles in the other.
- Two sides and a non-included angle in one of the triangles equal to two sides and a non-included angle in the other.
- Two angles and a non-corresponding side in one of the triangles equal to two angles and a non-corresponding side in the other.

Examples

The following pairs of triangles are ***congruent***:

a)

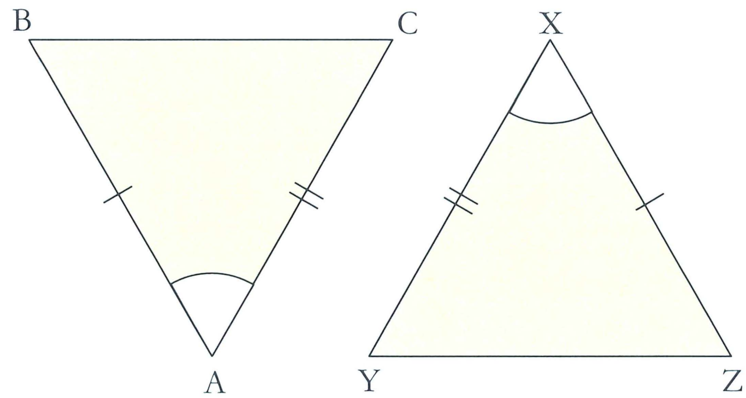


AB=XY (Three sides of ABC are equal to the three sides of XYZ).

BC=YZ

AC=XZ

b)



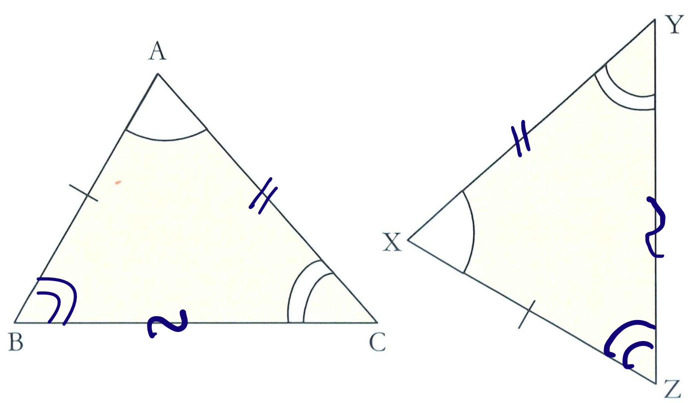
AB=XZ

AC=XY

∠BAC=∠YXZ

i.e. Two sides and the included angle in ABC equal to two sides and the included angle in XYZ)

c)



∠BAC=∠YXZ

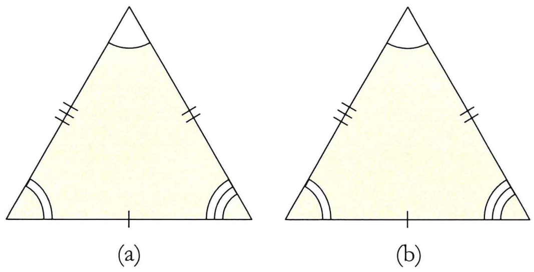
∠ACB=∠XYZ

AB=XZ

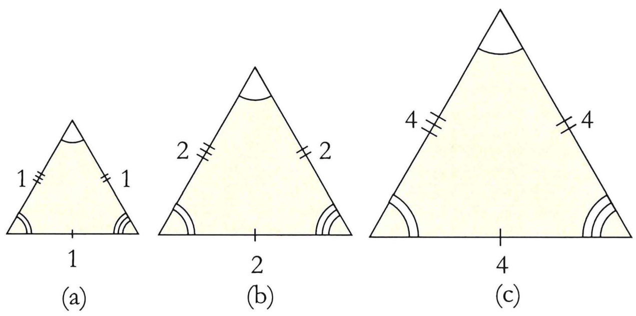
(Two angles and one side in ABC equal to two angles and a similarly placed side in XYZ)

Similar Triangles

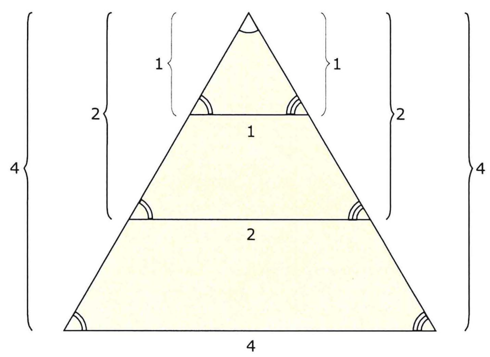
A pair of triangles are said to be ***similar*** if the sizes of the angles in one of them are identical to those in the other.



The above triangles are similar and congruent because the three sides in (a) are also the same size as those in (b). However it is possible to draw many different sized triangles with the same angles.



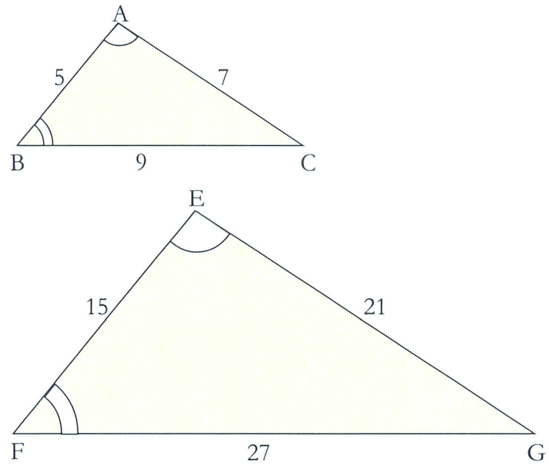
This is made clearer if we superimpose the three triangles.



The sides of the three triangles are clearly different but they are in the same ratio of 1:2 for (a) and (b), 2:4 (or 1:2) in the case of (b) and (c) and 1:4 in the case of (a) and (c).

So besides being equiangular, if two triangles are similar their corresponding sides are in the same ***ratio*** (but not equal in length necessarily).

Consider the following triangles:



AB:EF = 5 :15 = 3 : 1

AC:EG = 7 : 21 = 3 : 1

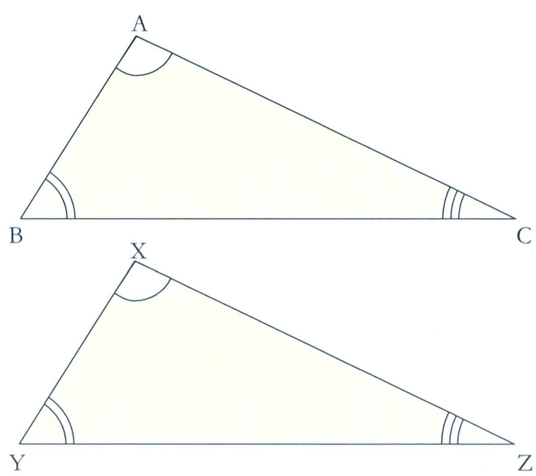
BC:FG = 9 : 27 = 3 : 1

Note: It does not matter which pair of sides you compare as long as they are ***corresponding sides***. By corresponding sides we mean the sides opposite to the equal angles. It is clear then that the triangles ABC and EFG are similar, because their corresponding sides are all in the same ratio of 3:1.

Calculations Involving Similar Triangles

If you are told that two triangles are equiangular then they are ***similar triangles***.

Example Triangles ABC and XYZ are equiangular.



$$\therefore \quad \frac{AB}{XY} = \frac{AC}{XZ} = \frac{BC}{YZ}$$

When solving problems on similar triangles, it helps if we write the two triangles with the equal angles underneath each other.

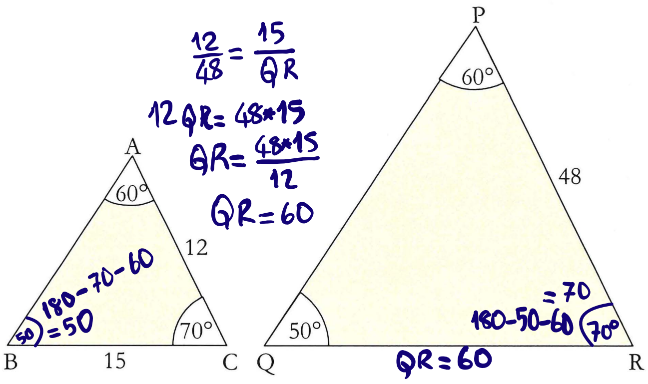
i.e. If we have two similar triangles ABC and PQR, where ∠A = ∠P, ∠B = ∠Q and ∠C = ∠R, we can write:-

$$\frac{ABC}{PQR}$$

The equation connecting the corresponding sides of the triangles is found by writing any two letters of the first triangle over the corresponding letters of the second one.

$$\frac{\overset{1\,2\,3}{ABC}}{\underset{1\,2\,3}{PQR}} = \frac{\overset{1\,2}{AB}}{\underset{1\,2}{PQ}} = \frac{\overset{2\,3}{BC}}{\underset{2\,3}{QR}} = \frac{\overset{1\,3}{AC}}{\underset{1\,3}{PR}}$$

Example 1 Find QR



In ΔABC, ∠ABC = 180° - (60° + 70°) = 50°

In ΔPQR, ∠PRQ = 180° - (60° + 50°) = 70°

Therefore Δ's ABC and PQR are equiangular (i.e. ***similar***) so we can write:

$$\frac{ABC}{PQR} = \frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$$

$$= \frac{BC}{QR} = \frac{AC}{PR}$$

We include the ratio with the two known sides and the ratio with the side we want. We leave out $\frac{AB}{PQ}$ because we don't know either.

$$= \frac{15}{QR} = \frac{12}{48}$$

$$12QR = 15 \text{ x } 48$$

$$QR = \frac{15 \text{ x } 48}{12}$$

$$= 60$$

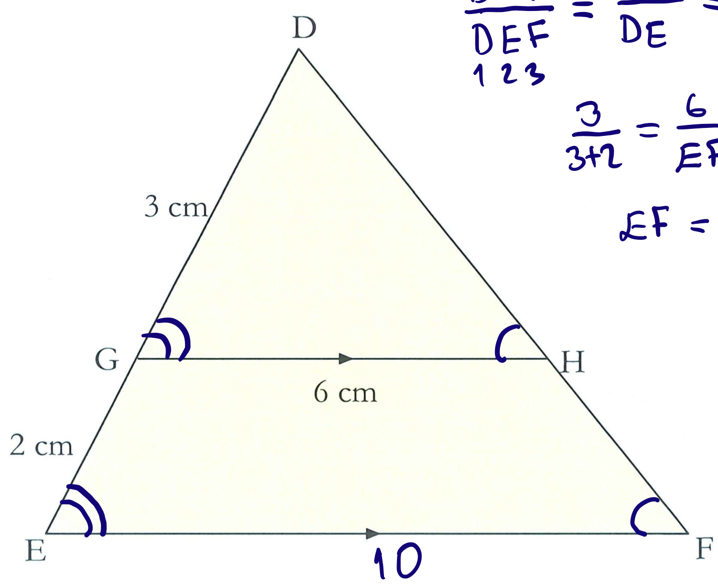
Example 2

In the following diagram GH is parallel to EF. If DG = 3cm, EG = 2cm and GH = 6cm, find the length of EF.

$$\frac{DGH}{DEF} = \frac{DG}{DE} = \frac{GH}{EF} = \frac{DH}{DF}$$

$$\frac{3}{3+2} = \frac{6}{EF}$$

$$EF = \frac{6*5}{3} = 10$$



The first thing to notice here is that we are not told that the triangles are similar, so we have to prove this.

In Δ's DGH and DEF
$\angle$GDH = $\angle$EDF (Common Angle)
$\angle$DGH = $\angle$DEF (Corresponding angles since GH parallel to EF)
$\angle$DHG = $\angle$DFE (Corresponding angles since GH parallel to EF)
Therefore Δ's DGH and DEF are similar (AAA)

$$\frac{DGH}{DEF} = \frac{DG}{DE} = \frac{GH}{EF} = \frac{DH}{DF}$$

ie $\frac{3}{5} = \frac{6}{EF}$ [DH + DF are both unknown]

$$EF = \frac{6 \times 5}{3}$$

$$EF = 10$$

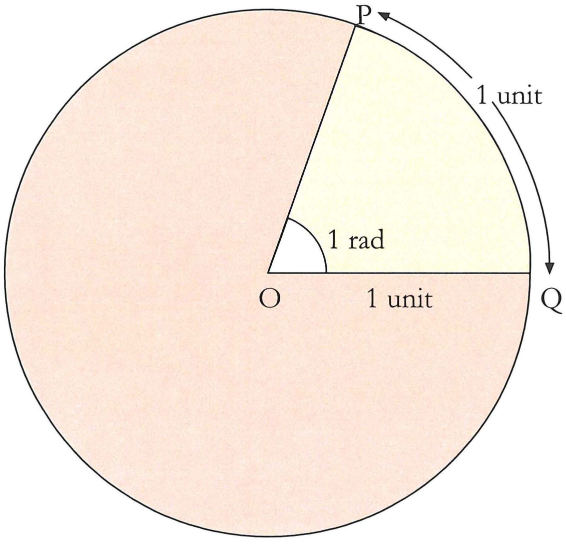
Check:
DG : DE = 3 : 5
GH : EF = 6 : 10 = 3 : 5

Note: (AAA) means (Angle, Angle, Angle) i.e. (condition for similarity of triangles).

Radians and Radian Measure

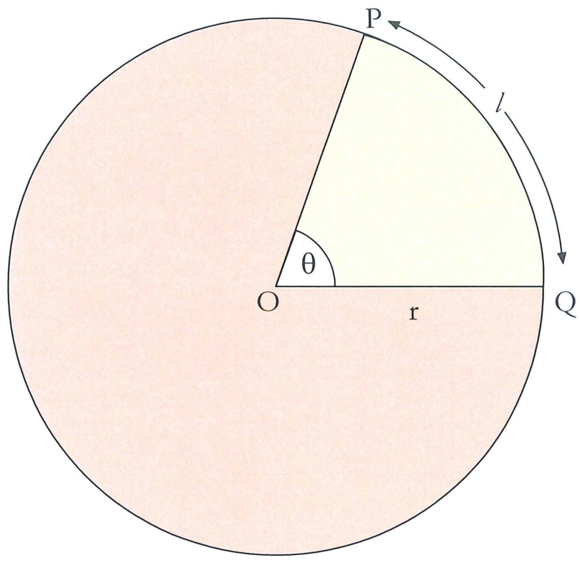
Normally angles are measured in degrees or parts of a degree. A radian is a larger unit that is used in trigonometry.

Definition. ***A radian is the angle subtended at the centre of a circle by an arc which is equal in length to the radius of the circle.***



In the diagram if the radius of the circle OQ is 1 unit and the length of the arc PQ is 1 unit then the angle $\angle$POQ is equal to 1 radian.

In general:



$$\text{Angle in Radians} = \frac{\text{length of arc (l)}}{\text{radius of circle (r)}}$$

$$\theta \text{ radians} = \frac{l}{r}$$

Cross multiply $l = r\theta$

The circumference of a circle is 2π units and therefore the angle at the centre of the circle θ subtended (formed) by the whole circumference is 2π radians. The angle at the centre is also 360°.

$$\text{So } 2\pi \text{ radians} = 360^\circ$$

$$\pi \text{ radians} = 180^\circ$$

$$1 \text{ radian} = \frac{180^\circ}{\pi}$$

$$\text{ie } 1 \text{ radian} = 57.3^\circ \text{ (1 dp)}$$

To Convert Degrees to Radians

$$\text{Since } 180^\circ = \pi \text{ radians}$$

$$\text{Then } 1^\circ = \frac{\pi \text{ radians}}{180}$$

$$\text{and } \boxed{\theta^\circ = \frac{\pi\theta}{180} \text{ radians}} \text{ — equation (1)}$$

Equation (1)

To Convert Radians to Degrees

Since π radians = 180°

Then 1 radian = $\dfrac{180°}{\pi}$

$$\theta \text{ rad} = \left(\frac{180 \times \theta}{\pi}\right)^{\circ} \quad \text{— equation (2)}$$

Equation (2)

Example 1

Convert (a) 30° (b) 60° (c) 90° to radians

From (1) above:

a) $30° = \dfrac{\pi \times 30}{180} = \dfrac{\pi}{6}$ radians

b) $60° = \dfrac{\pi \times 60}{180} = \dfrac{\pi}{3}$ radians

c) $90° = \dfrac{\pi \times 90}{180} = \dfrac{\pi}{2}$ radians

Example 2

Convert 36° 27' 13" to radians, giving your answer to 3 decimal places.

First of all, we must change the minutes and seconds into decimals of a degree.

Now since 60 seconds = 1 min and 60 minutes = 1°

Then there are: $60 \times 60 = 3600$ seconds in 1°.

So,

36°27'13" becomes $\left(36 + \dfrac{27}{60} + \dfrac{13}{3600}\right)° = 36.454°$

Now since $\theta° = \dfrac{\pi\theta}{180}$ radians

Then $36.454° = \dfrac{\pi \times 36.454}{180} = 0.636$ radians (3 dp)

Example 3

Convert 0.3628 radians into degrees, minutes and seconds.

$$\text{Now since } \theta \text{ radians} = \left(\frac{180 \times \theta}{\pi}\right)^\circ$$

$$\text{then } 0.3628 \text{ radians} = \frac{180 \times 0.3628}{\pi}$$

$$= 20.7869^\circ$$

If we take the decimal part and multiply by 60 we get

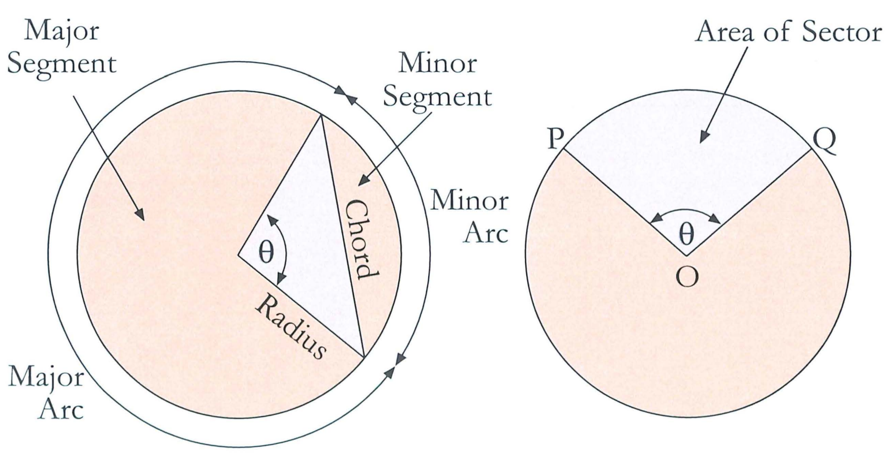
$0.7869 \times 60 = 47.214$ minutes.

And again taking the decimal part and multiplying by 60 we get

$0.214 \times 60 = 12.84$ or, 13 secs. (rounded)

So 0.3628 radians = 20° 47’ 13”

The Components of a Circle



Area of Circular Sector

$$\text{Area of circle} = \pi r^2 \text{ (subtended by } 360^\circ\text{)}$$

$$\text{Area of sector} = \frac{\pi r^2}{360} \text{ (subtended by } 1^\circ\text{)}$$

$$\text{Area of sector subtended by } \theta^\circ = \frac{\pi r^2 \theta}{360}$$

$$\text{now since } 360^\circ = 2\pi \text{ radians}$$

$$\text{Area of sector subtended by 1 radian} = \frac{\pi r^2}{2\pi}$$

$$\text{Area of sector subtended by } \theta \text{ radian} = \frac{\pi r^2 \theta}{2\pi} = \tfrac{1}{2} r^2 \theta$$

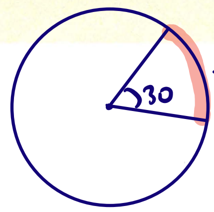
Summarising we have:

$$\text{Length of arc (l)} = \frac{2\pi r^2}{360} \times \theta^\circ = \frac{\pi r^2 \theta^\circ}{180} \qquad [\theta \text{ in degrees}]$$

$$\text{Length of arc (l)} = \frac{2\pi r}{2\pi} \times \theta^\circ = r\theta \qquad [\theta \text{ in radians}]$$

$$\text{Area of sector} = \frac{\pi r^2}{360^\circ} \times \theta = \frac{\pi r^2 \theta}{360} \qquad [\theta \text{ in degrees}]$$

$$\text{Area of sector} = \frac{\pi r^2}{2\pi} \times \theta = \tfrac{1}{2} r^2 \theta \qquad [\theta \text{ in radians}]$$



$\frac{30}{360} * 18.852$

$= 1.571$ m

Example 1

If the radius of a circle is r and θ is the angle subtended by an arc, find the length of the arc when r = 3m and θ = 30°. The circumconference = $2\pi r = 2 \times 3.142 \times 3$

$= 18.852$ m

$$\text{Length of arc (l)} = \frac{2\pi r \times \theta}{360} = \frac{2 \times 3.142 \times 3 \times 30^\circ}{360} = 1.57\text{m}$$

or since 1 radian = 57.3°

$$\text{then } \frac{1}{57.3} \text{ rads} = 1^\circ$$

$$\text{and } \frac{1}{57.3} \times 30 \text{ rads} = 30^\circ$$

i.e. Length of arc (l) = rθ

$$= 3 \times \frac{1}{57.3} \times 30$$

$$= 1.57\text{m}$$

Example 2

Find the area of the sector of a circle which has a radius 3cm and a sector angle of 60°

$$\text{Area} = \frac{\pi r^2\, 60}{360} = \frac{3.142 * 3^2 * 60}{360} = \frac{9 * 3.142}{6} \doteq 4.713 \text{ cm}^2$$

$$\text{Area of sector} = \frac{\pi r^2\theta}{360}$$

$$= \frac{3.142 \times 3 \times 3 \times 60°}{360}$$

$$= 4.71 \text{ cm}^2$$

or

since 1 radian = 57.3°

then

$$\frac{1}{57.3} \times 60 \text{ rads} = 60°$$

$$\text{Area of sector} = \tfrac{1}{2} r^2\theta$$

$$= \tfrac{1}{2} \times 3 \times 3 \times \frac{60}{57.3}$$

$$= 4.71 \text{ cm}^2$$

Example 3

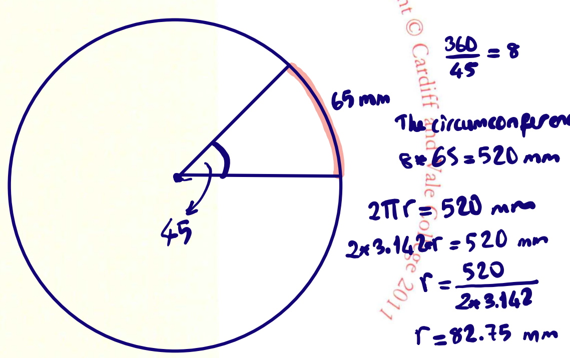
If an arc of 65mm long subtends an angle of 45° at the centre of the circle, find the radius of the circle.

$$\text{Length of arc (l)} = \frac{2\pi r\theta°}{360} = \frac{2 \times 3.142 \times r \times 45}{360}$$

i.e. $65 = 0.78r$

$$r = \frac{65}{0.7855}$$

$$r = 82.7 \text{ mm}$$



or again since 57.3° = 1 rad

$$1° = \frac{1}{57.3}$$

$$45° = \frac{1}{57.3} \times 45$$

$$= 0.785 \text{ radians}$$

$$\text{Length of arc (l)} = r\theta$$

$$r = \frac{l}{\theta}$$

$$r = \frac{65}{0.785}$$

$$r = 82.8 \text{ mm}$$

The answer differs slightly because of the approximations taken for π and the number of degrees in 1 radian.

Example 4

Find the area of the sector of a circle which has a radius of 78cm and a sector angle of 134° 42'.

Now $134° \ 42' = \left(134\frac{42}{60}\right)^{\circ} = 134.7°$

Area of sector $= \frac{\pi r^2\theta}{360} = \frac{3.142 \times 78 \times 78 \times 134.7}{360} = 7152.5 \text{ cm}^2$

or

Area of sector $= \frac{1}{2}r^2\theta$

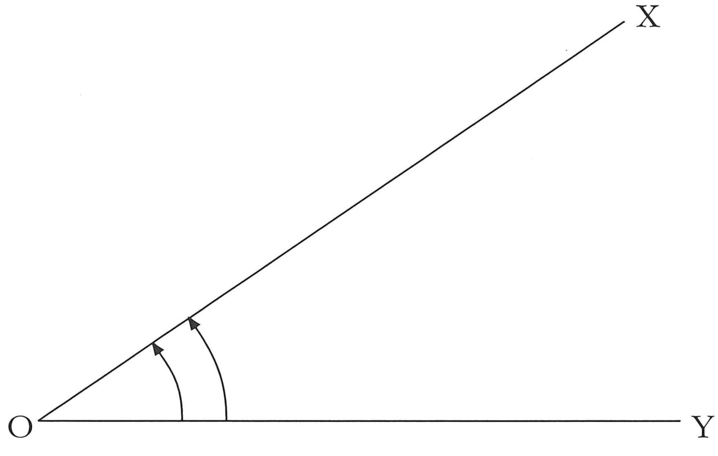
$= \frac{1}{2} \times 78 \times 78 \times \frac{134.7}{57.3}$

$= 7151.1 \text{ cm}^2$

Again a slight difference in answers, because of the approximations taken.

Angles

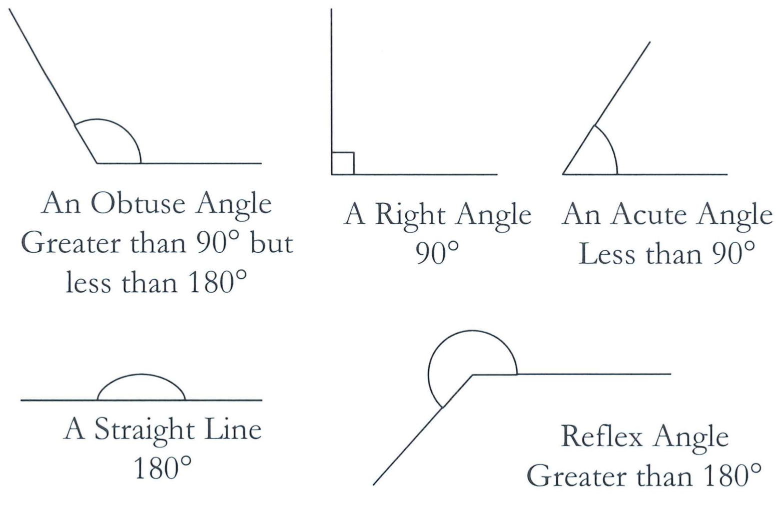
When two straight lines meet at a point they form an angle. The angle between X0 and 0Y is called the angle X0Y and is represented by $\angle$XOY.



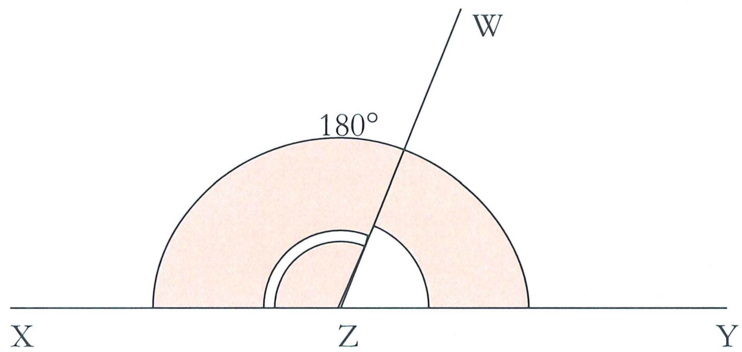
The actual size of the angle X0Y is a measure of how far the line 0X has been rotated anticlockwise from 0Y assuming at the start they were lying on top of each other.

If 0X continued to rotate it would eventually, after a full turn of 360°, come to rest on 0Y again. So a rotation through a full circle is 360° and that through a half-circle must be 180°.

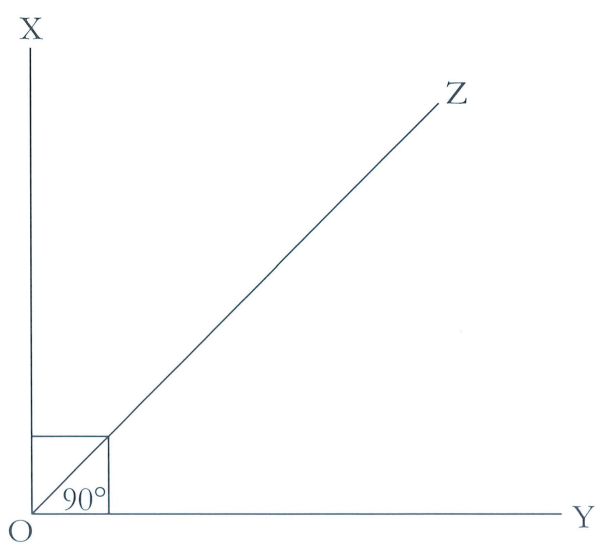
There are various types of angles.



If we consider a straight line XY and draw a second line WZ to it as shown, we get two angles formed: ∠WZX (***obtuse***) and ∠WZY (***acute***).

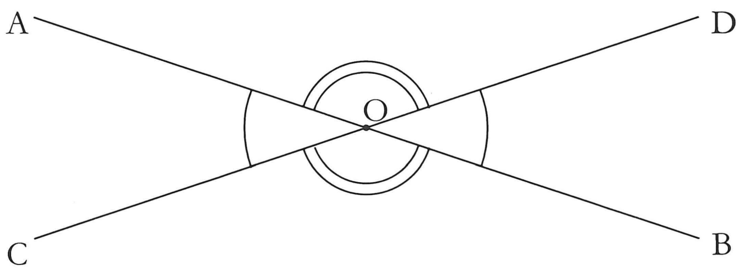


Since the angle on the straight line XY is 180° then ∠WZX + ∠WZY = 180° Angles of this type which add together to make 180° are called ***supplementary angles***.



In the above diagram, the right angle ∠X0Y has been split to form the angles ∠X0Z and ∠Z0Y. So ∠X0Z + ∠Z0Y = 90°. Angle of this type which add together to make 90° are called ***complementary angles***.

When two straight lines intersect, the opposite angles so formed are equal.



They are called ***vertically opposite angles***. So in the diagram above:

∠A0C = ∠D0B (vertically opposite angles)

and

∠A0D = ∠B0C (vertically opposite angles).

Examples

1. How many degrees are there in $\frac{2}{5}$ of a right angle?

Since a right angle = 90° then $\frac{2}{5}$ of a right angle = $\frac{2}{5} \times 90° = 36°$

2. How many degrees are there in 0.8 of a right angle?

Since a right angle = 90° then 0.8 of a right angle = $0.8 \times 90° = 72°$

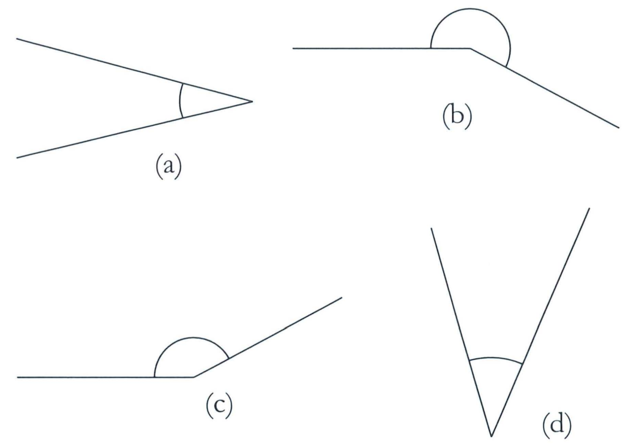
3. How many degrees are there in $\frac{4}{5}$ of a full turn?

Since a full turn = 360° then $\frac{4}{5} \times 360° = 288°$

4. How many degrees are there in 0.38 of a full turn?

Since a full turn = 360° then 0.38 of a full turn = $0.38 \times 360° = 136.8°$

5. State which of the following angles are acute, obtuse or reflex.



a) =acute b) =reflex c) = obtuse d) = acute

6. Two angles are supplementary. If one of them is 47°, what is the size of the other?

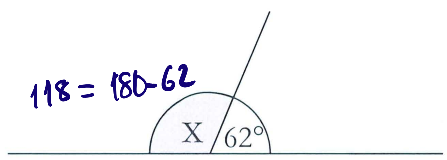
Supplementary angles add up to 180° ∴ if one is 47°, the other must be 180° - 47° = 133°

7. Two angles are complimentary. If one of them is 31° what is the size of the other?

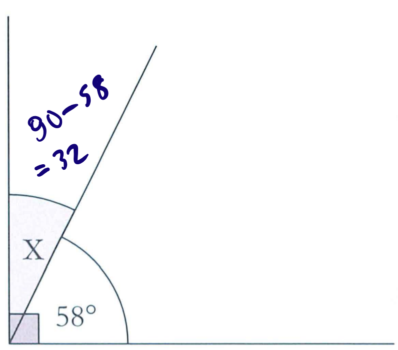
Complimentary angles add up to 90° ∴ if one is 31°, the other must be 90° - 31° = 59°.

8. Find the value of the angle x in each of the following:-

(a)



(b)



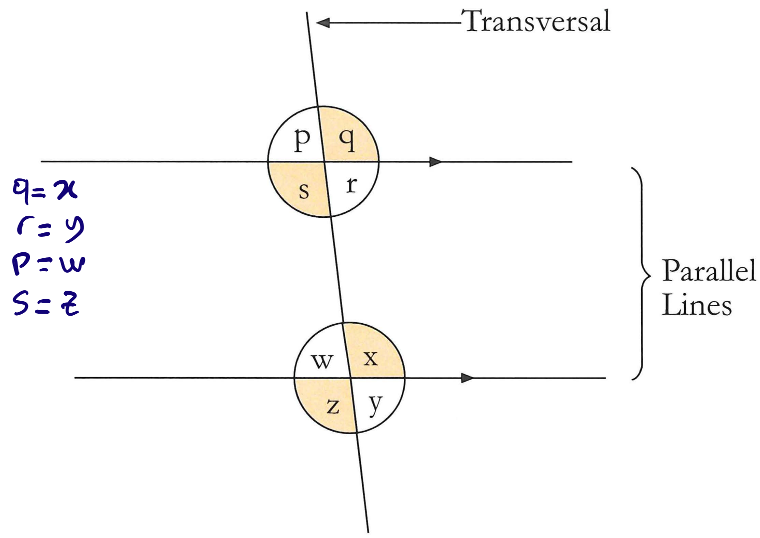
a) The two angles x and 62° lie on a straight line. The sum of the angles on a straight line is 180^0 ∴ x = 180° – 62° = 118°.

b) The two angles x and 58°, form a right angle between them. Since a right angle = 90° then x = 90° – 58° = 32°

Parallel Lines

Parallel lines are lines in the same plane that never meet irrespective of how far they are produced in either direction.

When two parallel lines are cut by a third line called a ***transversal*** as shown in the following diagram, then:-



1. The corresponding angles are equal. i.e. p = w; q = x; s = z; r = y

2. The alternate (Z angles) are equal. i.e. s = x; r = w

3. The interior angles add up to 180° (***always!***). In other words they are ***supplementary*** angles.

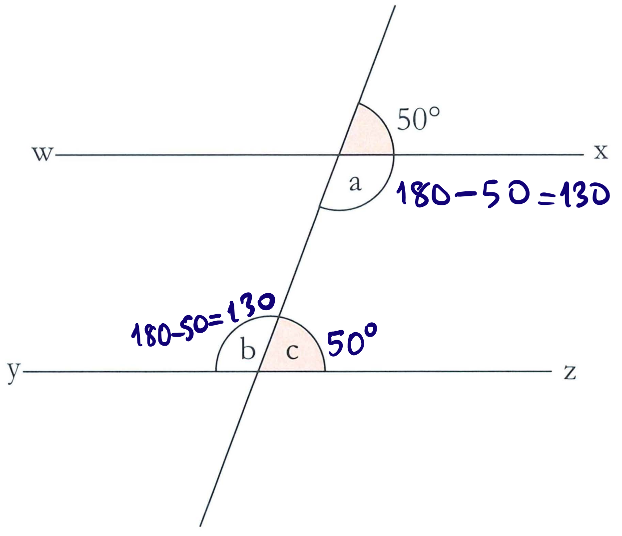
So s + w = 180° and r + x = 180°

The converse of the above is also true. If two straight lines are cut by a transversal then the two lines are parallel if any one of the following exists.

- Two angles in corresponding positions are equal.
- Two angles in alternate positions are equal.
- Two of the interior angles add up to 180° (i.e. are supplementary).

Example 1

In the following diagram, the lines WX and YZ are parallel. Find the values of the angles marked a, b, and c.



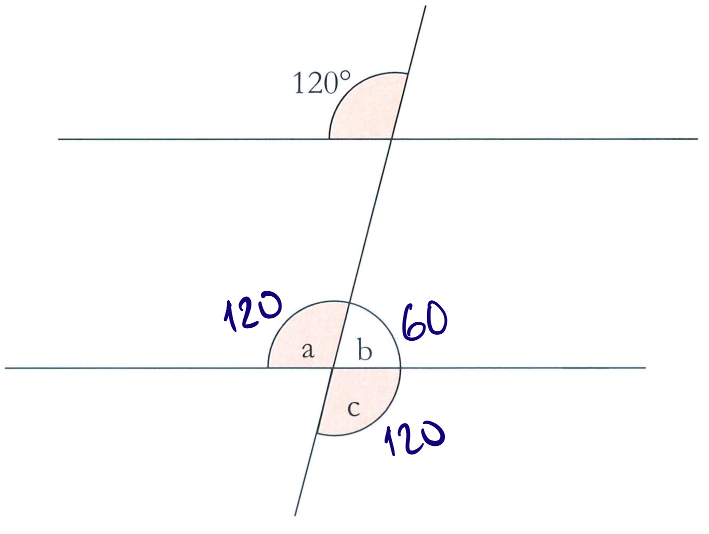
1) c = 50° (It is in the corresponding position to the angle 50° and the corresponding angles are equal).

2) a =180° - 50° = 130° (a and c are interior angles and therefore add up to 180°)

3) b = 130° (b is alternate to a, which = 130°, alternate angles are equal)

Example 2

Find the values of a, b and c from the following diagram.



a = 120° (a is in the corresponding position to the angle 120°)

c = 120° (c is vertically opposite to a and vertically opposite angles are equal)

b = 60° (a and b lie on a straight line ∴ a + b = 180°, so since a = 120°, b = 180° - 120° = 60°)