Example 5 $3(a - b) - 2(2a - 3b)$

$= 3a - 3b - 4a + 6b$

$= -a + 3b$

Example 6 $2x(x - 5) - x(x - 2) - 3x(x - 5)$

$= 2x^2 - 10x - x^2 + 2x - 3x^2 + 15x$

$= -2x^2 + 7x$

Factorisation

An expression such as $2x + 2y$ has the number 2 common to both terms.

i.e. $2x + 2y = 2(x + y)$ 2x will have two factors 2 and x

This is the reverse procedure of expanding brackets. The number 2 and the bracket $(x + y)$ are called ***the factors*** of $2x + 2y$. The easiest way to factorise an expression is to write out the expression in full breaking down the numbers into prime numbers (if they are not prime to start).

For example, $4 = 2 \times 2$ $6 = 2 \times 3$ $8 = 2 \times 2 \times 2$ $9 = 3 \times 3$ (and so on).

Example 1 Factorise $8p + 12q$

If we breakdown 8 and 12 into prime numbers and write out the expression in full we get:

$2 \times 2 \times 2 \times p + 2 \times 2 \times 3 \times q$

Now highlight numbers or letters which appear in ***both*** terms:

(2) $\times$ **(2)** $\times 2 \times p +$ **(2)** $\times$ **(2)** $\times 3 \times q$ $4 \times 2p + 4 \times 3q$

So **(2)** $\times$ **(2)** $= 4$ and this is placed outside the bracket. The unhighlighted parts which are left form the bracket. (In this case $2p + 3q$). These are the factors of $8p + 12q$.

$4 \times (2p + 3q)$

$$8p + 12q = 4(2p + 3q)$$

factors

Example 2 Factorise $9x^2 - 4x$

$= 3 \times 3 \times (x) \times x - 2 \times 2 \times (x)$ $9 \times x \times x - 4 \times x$

$= x(9x - 4)$ $\underbrace{x}_{\text{common factor}} \times (\underbrace{9x - 4}_{\text{Another factor}})$

Example 3 Factorise $6a - 6b$ → 6 is a common factor

$= (2) \times (3) \times a - (2) \times (3) \times b$ so $6 \times (a - b)$

$= 6(a - b)$

Example 4 Factorise $3a + 6a^2$

$= \boxed{(3) \times (a)} + 2 \times \boxed{(3) \times (a)} \times a$ $3a \times (1 + 2a)$

$= 3a(1 + 2a)$

Example 5 Factorise $3x - 6y + 12z$

$= (3) \times x - 2 \times (3) \times y + 2 \times 2 \times (3) \times z$ $3 \times (x - 2y + 4z)$

$= 3(x - 2y + 4z) = 3x - 6y + 12z$ ✓

Note: A good check to see whether your factors are correct is to multiply them together to see if you get back to the original expression.

Factorising by Grouping

To factorise an expression which has ***four terms***, group the terms in ***two pairs*** so that each pair has a common factor.

It should be noted that this method can only be used when there is a common factor.

Example 1 Factorise $\boxed{pr + qr} + \boxed{ps + qs} = r\boxed{(p+q)} + s\boxed{(p+q)}$

$= r(p + q) + s(p + q)$ $= (p+q)(r+s)$

Now the common bracket $(p + q)$ becomes a common factor and the other factor is $r + s$ which is also placed in brackets. $(p + q)(r + s)$

Example 2 Factorise $\boxed{ax + ay} - \boxed{bx - by}$ $= a\boxed{(x+y)} - b\boxed{(x+y)}$

$= a(x + y) - b(x + y)$ $= (x+y)(a-b)$

Note the change of sign when the second bracket is formed. Finally we have $(x + y)(a - b)$ Second way:

$ax+ay-bx-by = \boxed{ax-bx} + \boxed{ay-by} = x(a-b)+y(a-b)$
$=(a-b)(x+y)$

Use the Four Arithmetic Processes on Algebraic Fractions

When we add two fractions together in arithmetic the procedure is:

- Find the lowest common denominator of the two fractions to be added.
- Express each of the fractions with this common denominator.
- Add the two numerators and put the answer over this common denominator.

We follow exactly the same procedure when we add two algebraic fractions together.

If we compare the two:-

$\frac{3}{4} \times \frac{5}{5} + \frac{1}{5} \times \frac{4}{4} = \frac{15}{20} + \frac{4}{20}$ $\frac{15+4}{20} = \frac{19}{20}$

$$\frac{3}{4} + \frac{1}{5} = \frac{5(3) + 4(1)}{20} = \frac{19}{20}$$

$$\frac{x}{4} + \frac{x}{5} = \frac{5(x) + 4(x)}{20} = \frac{9x}{20}$$

$\frac{x}{4} \times \frac{5}{5} + \frac{x}{5} \times \frac{4}{4} = \frac{5x}{20} + \frac{4x}{20}$
$= \frac{5x+4x}{20} = \frac{9x}{20}$

So if we have an expression such as:

$\frac{p}{q} \times \frac{s}{s} + \frac{r}{s} \times \frac{q}{q} = \frac{ps}{qs} + \frac{rq}{qs} = \frac{ps+rq}{qs}$

$$\frac{p}{q} + \frac{r}{s}$$

Firstly we have to find the lowest common denominator. This will be the lowest common multiple of q and s, i.e. (qs). Each fraction is then expressed with this denominator.

$$\frac{p}{q} + \frac{r}{s} = \frac{p(s) + r(q)}{qs} = \frac{ps + rq}{qs}$$

Example 1

Express as a single fraction $\frac{3}{p} + \frac{5}{p}$ $= \frac{3+5}{p} = \frac{8}{p}$

Here the fractions have a common denominator p.

$$\frac{3}{p} + \frac{5}{p} = \frac{3(1) + 5(1)}{p} = \frac{8}{p}$$

Example 2

Express as a single fraction $\frac{x}{y^2} + \frac{1}{yz}$

$\frac{x}{y^2} \times \frac{z}{z} + \frac{1}{yz} \times \frac{y}{y} = \frac{xz}{y^2z} + \frac{y}{y^2z}$

$= \frac{xz+y}{y^2z}$

The lowest common denominator is the LCM of y and z which is y^2z

$$\frac{x(z) + y}{y^2z} = \frac{xz + y}{y^2z}$$

For subtraction of algebraic fractions, the method is similar to addition except that after expressing each of the fractions with a common denominator, the numerators are subtracted.

Example 1

$\frac{x}{3} \times \frac{4}{4} - \frac{x}{4} \times \frac{3}{3} = \frac{4x}{12} - \frac{3x}{12}$

$$\frac{x}{3} - \frac{x}{4} = \frac{4(x) - 3(x)}{12} = \frac{x}{12}$$

$= \frac{4x-3x}{12} = \frac{x}{12}$

Example 2

$\frac{2y}{5} \times \frac{3}{3} - \frac{y}{3} \times \frac{5}{5} = \frac{6y}{15} - \frac{5y}{15}$

$$\frac{2y}{5} - \frac{y}{3} = \frac{3(2y) - 5(y)}{15} = \frac{6y - 5y}{15} = \frac{y}{15}$$

$= \frac{6y-5y}{15} = \frac{y}{15}$

Example 3

$\frac{4}{5y}\times\frac{3}{3}-\frac{1}{3y}\times\frac{5}{5}=\frac{12}{15y}-\frac{5}{15y}=\frac{12-5}{15y}$

$=\frac{7}{15y}$

$$\frac{4}{5y}-\frac{1}{3y}=\frac{3(4)-5(-1)}{15y}=\frac{12-5}{15y}=\frac{7}{15y}$$

It is possible to have a mixture of addition and subtraction. In these cases just be careful with the signs in front of the terms.

Example 4

$\frac{x}{5}+\frac{x}{2}=\frac{x}{5}\times\frac{2}{2}+\frac{x}{2}\times\frac{5}{5}$

$=\frac{2x}{10}+\frac{5x}{10}$

$=\frac{7x}{10}$

$$\frac{x}{5}+\frac{x}{2}-\frac{x}{3}=\frac{6(x)+15(x)-10(x)}{30}=\frac{11x}{30}$$

$=\frac{7x}{10}-\frac{x}{3}=\frac{7x}{10}\times\frac{3}{3}-\frac{x}{3}\times\frac{10}{10}=\frac{21x}{30}-\frac{10x}{30}$

$=\frac{21x-10x}{30}=\frac{11x}{30}$

Example 5

$\frac{2}{7y}-\frac{1}{4y}=\frac{2}{7y}\times\frac{4}{4}-\frac{1}{4y}\times\frac{7}{7}$

$=\frac{8}{28y}-\frac{7}{28y}=\frac{8-7}{28y}$

$=\frac{1}{28y}$

$$\frac{1}{3y}+\boxed{\frac{2}{7y}-\frac{1}{4y}}=\frac{28(1)+12(2)-21(1)}{84y}=\frac{31}{84y}$$

$\frac{1}{3y}+\frac{1}{28y}=\frac{1}{3y}\times\frac{28}{28}+\frac{1}{28y}\times\frac{3}{3}=\frac{28}{84y}+\frac{3}{81y}=\frac{28+3}{81y}$

$=\boxed{\frac{31}{81y}}$

Multiplication and Division of Algebraic Fractions

When multiplying algebraic fractions then:

- Multiply the numerators (tops) together.
- Multiply the denominators (bottoms) together.
- Place the numerator product over the denominator product and simplify by cancelling down where possible.

Examples

(1) $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$

(2) $\frac{a^2b}{c}\times\frac{d}{e}=\frac{a^2bd}{ce}$

(3) $\frac{4a^2\cancel{b}}{\cancel{c}}\times\frac{3\cancel{c}d}{\cancel{b}}=\frac{12\times a\times a\times\cancel{b}\times\cancel{c}\times c\times d}{\cancel{c}\times\cancel{b}}=12a^2cd$

$\frac{4a^2\times 3cd}{1}$

$\boxed{12a^2cd}$

(4) $\frac{\cancel{6ab}}{\cancel{3xy}}\times\frac{7x^2y}{\cancel{2ab^3}}=\frac{\cancel{6}\times\cancel{a}\times\cancel{b}\times 7\times\cancel{x}\times x\times\cancel{y}}{\cancel{3}\times\cancel{x}\times\cancel{y}\times\cancel{2}\times\cancel{a}\times\cancel{b}\times b\times b}=\frac{7x}{b^2}$

$\frac{7xy}{b^2}$

Numbers (factors) which are common to top and bottom may be cancelled. This is equivalent to dividing top and bottom by the same number or symbol.

$$\frac{a}{4y^2} \qquad (5) \quad \frac{10a^2b}{16xy^3} \times \frac{2xy}{5ab} = \frac{\cancel{20} \times \cancel{a} \times a \times \cancel{b} \times \cancel{x} \times \cancel{y}}{\cancel{80} \times \cancel{x} \times x \times \cancel{y} \times y \times y \times \cancel{a} \times \cancel{b}} = \frac{a}{4y^2}$$

$$(6) \quad \frac{3ab}{c} \times \frac{bd}{2e} \times \frac{3cd}{ab^2} = \frac{9 \times \cancel{a} \times \cancel{b} \times \cancel{b} \times d \times \cancel{c} \times d}{\cancel{c} \times 2 \times e \times \cancel{a} \times \cancel{b} \times \cancel{b}} = \frac{9d^2}{2e}$$

$$\frac{9d^2}{2e}$$

Division of Algebraic Fractions

As for normal fractions, when the algebraic fractions have a division (÷) sign between them, then remember to change the sign to a multiplication (×) and invert, (i.e. turn upside down) the fraction you are dividing by.

Example 1

$$(1) \quad \frac{xy^3}{x^2y} \div \frac{x^2}{xy} = \frac{xy^3}{x^2y} \times \frac{xy}{x^2} = \frac{y^3}{x^2}$$

$$= \frac{xy^3}{x^2y} \times \frac{xy}{x^2}$$

$$= \frac{\cancel{x} \times \cancel{y} \times y \times y \times \cancel{x} \times y}{\cancel{x} \times \cancel{x} \times \cancel{y} \times x \times x}$$

$$= \frac{y^3}{x^2}$$

Example 2

$$(2) \quad \frac{12pq}{10rs} \div \frac{4p^2}{9qs} = \frac{12pq}{10rs} \times \frac{9qs}{4p^2} = \frac{27q^2}{10rp}$$

$$= \frac{12pq}{10rs} \times \frac{9qs}{4p^2}$$

$$= \frac{12 \times \cancel{p} \times q \times 9 \times q \times \cancel{s}}{10 \times r \times \cancel{s} \times 4 \times \cancel{p} \times p}$$

$$= \frac{108q^2}{40rp}$$

$$= \frac{27q^2}{10rp} \quad \left(\text{ie. Dividing top and bottom by 4}\right)$$

Example 3

(3) $\dfrac{6pq}{10rs} \div \dfrac{p^2}{15s^2} = \dfrac{\overset{9\cancel{0}}{\cancel{6}}\cancel{p}q}{\cancel{10}r\cancel{s}} \times \dfrac{\overset{s}{15\cancel{s^2}}}{\underset{p}{\cancel{p^2}}} = \dfrac{9s}{rp}$

$$= \frac{6pq}{10rs} \text{ x } \frac{15s^2}{p^2}$$

$$= \frac{6 \text{ x } \overset{1}{\cancel{p}} \text{ x } q \text{ x } 15 \text{ x } \overset{1}{\cancel{s}} \text{ x } s}{10 \text{ x } r \text{ x } \underset{1}{\cancel{s}} \text{ x } \underset{1}{\cancel{p}} \text{ x } p}$$

$$= \frac{90qs}{10rp}$$

$$= \frac{9qs}{rp} \quad \left(\begin{array}{c}\text{ie. Dividing top and} \\ \text{bottom by 10}\end{array}\right)$$

Example 4

(4) $\dfrac{2a^2b}{3zy^2} \text{ x } \dfrac{6z^2y}{8ab} \div \dfrac{ab^2}{4zy}$ work it out yourself in the same way.

$$= \frac{2a^2b}{3zy^2} \text{ x } \frac{6z^2y}{8ab} \text{ x } \frac{4zy}{ab^2}$$

$$= \frac{\overset{2}{\cancel{48}} \text{ x } \overset{1}{\cancel{a}} \text{ x } \overset{1}{\cancel{a}} \text{ x } \overset{1}{\cancel{b}} \text{ x } \cancel{z} \text{ x } z \text{ x } \overset{1}{\cancel{y}} \text{ x } z \text{ x } \overset{1}{\cancel{y}}}{\underset{1}{\cancel{24}} \text{ x } \underset{1}{\cancel{z}} \text{ x } \underset{1}{\cancel{y}} \text{ x } \underset{1}{\cancel{y}} \text{ x } \underset{1}{\cancel{a}} \text{ x } b \text{ x } \underset{1}{\cancel{a}} \text{ x } \underset{1}{\cancel{b}} \text{ x } b}$$

$$= \frac{2z^2}{b^2}$$

Equations

An equation is a statement of equality. In very simple terms $3 + 2 = 5$ is an equation. The equations that we will look at have some of the numbers replaced by a letter or symbol. They are called algebraic equations. What we have to do is to find the value of the letter or symbol. This process is called solving the equation. Once we have found a value for the letter or symbol we can substitute it in the original equation to see if the sides balance. If they do, the equation is said to be satisfied by the value, and the value is called the solution of the equation.

Linear Equations

These contain only the first power of the unknown quantity. For example, only x or a numerical multiple of x and not x^2 or x^3. e.g. $5x + 3 = 13$

In the process of solving an equation, the equality of both sides of it must be maintained and so whatever we do to one side of the equation we must do exactly the same to the other.

Equations That Require Multiplication and/or Division

Example 1

Solve the equation

$\frac{x}{5} = \frac{2}{1}$

$x \times 1 = 5 \times 2$

$\boxed{x = 10}$

$$\frac{x}{5} = 2$$

$\cancel{5} \times \frac{x}{\cancel{5}} = 2 \times 5$

$\boxed{x = 10}$

If we multiply both sides by 5, we get

$$\frac{x}{\cancel{5}} \times \cancel{5} = 2 \times 5$$
$$x = 10$$

Check: $\frac{x}{5} = 2$

ie. $\frac{10}{5} = 2$

The process of cross multiplication achieves the same result.

We know that

$$\frac{6}{3} = \frac{4}{2}$$

If we cross multiply : top left × bottom right = bottom left × top right, we obtain $6 \times 2 = 3 \times 4$ which is true. So, in the same way:

$$\frac{x}{5} = \frac{2}{1}$$
$$x = 10$$

Example 2 Solve the equation 3x = 15

If we divide both sides by 3, we get:

$$\frac{3x}{3} = \frac{15}{3}$$

$$x = 5$$

Check $3x = 15$

When $x = 5$

$$3 \times 5 = 15$$

Some equations involve both of these processes.

For example, solve

$$\frac{5}{a} = 5$$

i.e. $\frac{5}{a} = \frac{5}{1}$

$\frac{5}{a} = \frac{5}{1}$

$5a = 5$

$\frac{5a}{5} = \frac{5}{5} \Rightarrow \boxed{a = 1}$

Cross multiply $5a = 5$

Dividing both sides by 5: $\frac{5a}{5} = \frac{5}{5}$

$a = 1$

Equations That Require Addition and/or Subtraction

Example 1 $x + 3 = 8$

If we subtract 3 from both sides we get: $x + 3 - 3 = 8 - 3$ therefore $x = 5$

Example 2 $x - 5 = 12$

If we add 5 to both sides we get: $x - 5 + 5 = 12 + 5$ therefore $x = 17$

In the examples above the same result can be achieved by transferring the numbers from the ***left hand side*** of the equals to the ***right hand side***, and changing the sign in front of them.

ie. $x + 3 = 8$

$x = 8 - 3$

$x = 5$

and $x - 5 = 12$

$x = 12 + 5$

$x = 17$

Movement of terms from one side of the equals to the other in either direction can be made <u>but</u> remember ***if you change side, then change sign***.

Equations That Contain the Unknown Quantity on Both Sides

In this type, we collect all terms containing the unknown quantity on one side of the equation (usually the left) and numbers on the other side (usually the right).

e.g. $5x + 3 = 2x + 15$

$5x – 2x = 15 – 3$

$3x = 12$ (transferring terms and changing signs)

$x = 4$

Example 1

$2x - 5 = 4x + 7$

$2x - 4x = 7+5$

$-2x = 12$ $\quad \frac{\cancel{-2}x}{\cancel{-2}} = \frac{12}{-2}$

$x = -6$ $\quad \boxed{x = -6}$

Check: $2(-6) - 5 = -17$ LHS

$4(-6) + 7 = -17$ RHS

Example 2

$3x - 7 = 5x - 9$ $\Rightarrow 9 - 7 = 5x - 3x$

$2 = 2x$

$\boxed{x = 1}$

$3x - 5x = -9 + 7$

$-2x = -2$

$x = 1$

Check: $3(1) - 7 = -4$ LHS

$5(1) - 9 = -4$ RHS

To see if our solutions are correct, we can check by substituting the value found back into both sides of the original equation, to see if the sides balance.

Equations That Contain Brackets

In equations of this type, the brackets must be expanded first and then we proceed as previously.

Example 1

$$3(2x + 5) = 27$$

$$6x + 15 = 27$$

$$6x = 27 - 15$$

$$6x = 12$$

$$x = 2$$

Check: $3(2(2) + 5) = 27$

Example 2

$$3(x + 5) + 2(x - 3) = 24$$

$$3x + 15 + 2x - 6 = 24$$

$$3x + 2x = 24 - 15 + 6$$

$$5x = 15$$

$$x = 3$$

Check: $3(3+5) + 2(3 - 3) = 24 + 0 = 24$

Equations Containing Fractions

When an equation contains fractions, the procedure is:

1. Find the LCM of the denominators.
2. Multiply each of the terms by this LCM.
3. Simplify and proceed as before.

Example 1

$$\frac{x}{3} + \frac{4}{5} = \frac{2x}{3} - 4 \Rightarrow \frac{4}{5} + 4 = \frac{2x}{3} - \frac{x}{3}$$

$$\frac{4 \times 5 + 4}{5} = \frac{x}{3}$$

The LCM of the denominators is 15. Multiplying each term by this we get:

$$\frac{24}{5} \times \frac{x}{3}$$

$$72 = 5x \Rightarrow \boxed{x = \frac{72}{5}}$$

$$x = 14.4$$

$$\frac{x}{\cancel{3}_1} \times \cancel{15}^5 + \frac{4}{\cancel{5}_1} \times \cancel{15}^3 = \frac{2x}{\cancel{3}_1} \times \cancel{15}^5 - 4 \times 15$$

$$x \times 5 + 4 \times 3 = 2x \times 5 - 4 \times 15$$

$$5x + 12 = 10x - 60$$

$$5x - 10x = -60 - 12$$

$$-5x = -72$$

$$x = 14.4$$

Example 2

$$\frac{x}{5} - \frac{x}{3} = 4 \Rightarrow \frac{x}{5} \times \frac{3}{3} - \frac{x}{3} \times \frac{5}{5} = 4$$

$$\frac{3x}{15} - \frac{5x}{15} = 4 \Rightarrow \frac{-2x}{15} \cancel{\times} \frac{4}{1}$$

The LCM of the denominators is 15. Multiplying each term by this we get:

$$\frac{x}{\cancel{5}_1} \times \cancel{15}^3 - \frac{x}{\cancel{3}_1} \times \cancel{15}^5 = 4 \times 15$$

$$3x - 5x = 60$$

$$-2x = 60$$

$$x = -30$$

$$-2x = 60$$

$$\boxed{x = -30}$$

Construction and Solving of Linear Equations

Many problems are solved more quickly if they are first written in algebraic terms.

To construct simple or linear equations:-

- Let the quantity to be found be represented by a letter. x is usually used.
- Using the given data, make up an equation which conforms to it.
- Before proceeding check that both sides of the expression are in the same units.

Example 1

Find three consecutive whole numbers whose sum is 66.

Let the first number be x. Since the numbers run consecutively, the second number must be x+1 and the third number x + 2.

So we have $\quad x + \underbrace{x+1}_{\text{second}} + \underbrace{x+2}_{\text{third}} = 66$

first

$$3x + 3 = 66$$

$$3x = 66 - 3$$

$$3x = 63$$

$$\therefore \quad x = 21$$

Example: Find 5 consecutive whole numbers whose sum is 155

$x + (x+1) + (x+2) + (x+3) + (x+4) = 155$

$5x + 1 + 2 + 3 + 4 = 155$

$5x + 10 = 155$

$5x = 155 - 10$

$5x = 145 \Rightarrow \boxed{x = 29}$

The first number x is 21, the second x + 1 is 22 and the third x + 2 is 23.

Check: 21 + 22 + 23 = 66

Example 2

When a number is trebled and 7 added, the result is 49. What is the number?

Let the number = x If we treble it we have 3 × x, i.e. 3x

So we have $\quad 3x + 7 = 49$

$$3x = 49 - 7$$

$$3x = 42$$

$$\therefore \quad x = 14$$

Check: 3(14) + 7 = 49

Example 3

The sides of a rectangle are x cm, (x + 3)cm, (x – 4)cm and (x + 6) cm. Find the lengths of the four sides if the perimeter of the rectangle is 73cm.

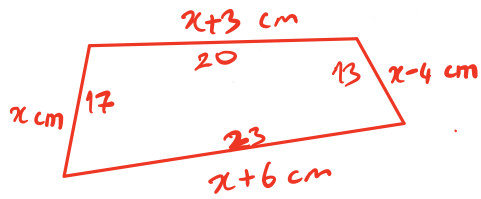
We have $\quad x + x + 3 + x - 4 + x + 6 = 73$

$$4x + 5 = 73$$

$$4x = 73 - 5$$

$$4x = 68$$

$$x = 17 \text{ cm}$$



$x + (x+3) + (x-4) + (x+6) = 73$

$4x + 5 = 73$

$4x = 68 \Rightarrow \boxed{x = 17}$

So, the four sides are 17cm, 20cm, 13cm and 23cm.

$x = 17$

$x + 3 = 20$

$x - 4 = 13$

$x + 6 = 23$

Check: 17 + 20 + 13 + 23 =73

Example 4

A man buys a total of 51 articles. Some are priced at 9p and others at 8p. If the total cost is £4.29, how many of each did he buy?

Let the number of 9p articles = x ∴ Let the number of 8p articles = 51 - x

£1 = 100 pence
£4.29 = 429 pence

So we have:

$9x + 8(51 - x) = 429p$ i.e. Units on both sides must be the same

$9x + 408 - 8x = 429$

$x = 429 - 408$

$x = 21$

Therefore we have 21 articles at 9p = 189p

And (51 - 21) i.e. 30 articles at 8p = 240p

∴ Total = 429p = £4.29

Powers and Indices

The product $a \times a \times a$ can be represented as a^3. The number 3, which tells us how many a's are multiplied together is called the ***index*** (plural - indices).

a^3 can be described as the third power of the algebraic base a.

When we multiply and divide powers of the same algebraic base, the indices combine according to certain rules.

Rule 1

When multiplying powers of the same algebraic base, ***add*** the indices.

If: $a^3 = a \times a \times a$ and $a^4 = a \times a \times a \times a$

then $a^3 \times a^4 = (a \times a \times a) \times (a \times a \times a \times a) = a^7$

In other words: $a^3 \times a^4 = a^{3+4} = a^7$

In general we may write:

$$a^m \times a^n = a^{m+n}$$

Further examples

$$y^2 \times y^6 = y^{2+6} = y^8$$

$$p^6 \times p^{12} = p^{6+12} = p^{18} \quad \text{(and so on).}$$

It is important to note that they must be ***powers of the same algebraic base***. The rule applies to ordinary numbers as well as algebraic symbols.

e.g. $\quad 2^3 \times 2^4 = 2^{3+4} = 2^7 = 128$

but $\quad a^4 \times b^2$ does not equal ab^6(different bases)

similarly $\quad 2^4 \times 3^2$ does not equal 6^6(different bases)

Rule 2

When we divide powers of the same algebraic base, we ***subtract*** the index of the denominator (bottom) from the index of the numerator (top).

$$\frac{a^7}{a^3} = \frac{\cancel{a}^{1} \text{ x } \cancel{a}^{1} \text{ x } \cancel{a}^{1} \text{ x a x a x a x a}}{\cancel{a}_{1} \text{ x } \cancel{a}_{1} \text{ x } \cancel{a}_{1}} = \frac{a^4}{1} = a^4$$

using rule $\quad a^{7-3}$

$$\boxed{\frac{a^n}{a^m} = a^{n-m}}$$

Examples

$$a^6 \div a^2 = \frac{a^6}{a^2} = a^{6-2} = a^4$$

$$x^7 \div x^3 = \frac{x^7}{x^3} = x^{7-3} = x^4$$

Similarly, when the base is a number we have:

$$2^6 \div 2^3 = \frac{2^6}{2^3} = 2^{6-3} = 2^3 = 8$$

In general:

$$\frac{a^m}{a^n} = a^{m-n}$$

There are instances when we combine rules (1) and (2)

For example,

$$\frac{x^5 \text{ x } x^2 \text{ x } x^7}{x^4 \text{ x } x^6} \overset{(1)}{=} \frac{x^{14}}{x^{10}} \overset{(2)}{=} x^{14-10} = x^4$$

Rule 3

When the index of the numerator is less than the index of the denominator

For example, $a^2 \div a^4$ we have:

$$a^2 \div a^4 = \frac{a^2}{a^4} = \frac{\cancel{a}^{1} \times \cancel{a}^{1}}{\cancel{a}_{1} \times \cancel{a}_{1} \times a \times a} = \frac{1}{a^2}$$ $= a^{2-4} = a^{-2} = \frac{1}{a^2}$

However, when we apply the division rule we obtain:

$$\frac{a^2}{a^4} = a^{2-4} = a^{-2}$$

Both these answers are correct, so:

$$a^{-2} = \frac{1}{a^2}$$

So a negative index indicates the reciprocal of a quantity.

In general:

$$a^{-m} = \frac{1}{a^m}$$

Rule 4

When the indices of the numerator and denominator are the same.

For example:

$$\frac{a^5}{a^5}$$

We have:

$$a^5 \div a^5 = \frac{a^5}{a^5} = \frac{\cancel{a}^{1} \times \cancel{a}^{1} \times \cancel{a}^{1} \times \cancel{a}^{1} \times \cancel{a}^{1}}{\cancel{a}_{1} \times \cancel{a}_{1} \times \cancel{a}_{1} \times \cancel{a}_{1} \times \cancel{a}_{1}} = \frac{1}{1} = 1$$

However when we apply the division rule we obtain:

$$\frac{a^5}{a^5} = a^{5-5} = a^0$$ except for $a=0$
so 0^0 is unknown.

Again, both these answers are correct so: $a^0 = 1$

So ***any*** quantity raised to the power of 0 is equal to 1.

In general:

$$a^0 = 1$$

Examples:

$$x^0 = 1,\ 7^0 = 1,\ (0.375)^0 = 1,\ \left(\frac{3}{5}\right)^0 = 1$$

Rule 5

When we raise a power of a quantity to another power, for example, $(x^3)^2$ this means: $(x^3)^2 = x^3 \times x^3 = x^{3+3} = x^6$ (from rule 1)

The same result can be obtained by multiplying the indices $(x^3)^2 = x^{3 \times 2} = x^6$

But take care when we have say $(2x^2)^3$ we must remember that 2 is the same as 2^1.

$(x^a)^b = x^{ab}$

So: $(2x^2)^3 = 2^{1 \times 3} x^{2 \times 3}$

$= 2^3 x^6$

$(2x^2)^3 = 2^3 x^{2\times 3} = \boxed{8x^6}$

$= 8x^6$

$\boxed{(x^a y^b)^c = x^{ac} y^{bc}}$

Further Examples

$(x^2y^4)^5 = x^{2 \times 5} y^{4 \times 5} = x^{10}y^{20}$ → $x^{2\times5} y^{4\times5} = x^{10}y^{20}$

$\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$

$(xy^3)^2 = x^{1 \times 2} y^{3 \times 2} = x^2y^6$ → x^2y^6

$\frac{81x^4}{y^8} = \frac{3^4x^4}{y^{2\times4}}$ ←

$$\left(\frac{3x}{y^2}\right)^4 = \frac{3^{1 \times 4} x^{1 \times 4}}{y^{2 \times 4}} = \frac{3^4 x^4}{y^8} = \frac{81x^4}{y^8}$$

So when raising the power of a quantity to a power, we ***multiply*** the indices.

In general:

$$\boxed{(a^m)^n = a^{m \times n}}$$

Rule 6

Fractional indices are also a possibility

From rule (1): a^1 may be written as $a^{1/2} \times a^{1/2} = a^{\frac{1}{2}+\frac{1}{2}} = a^1$

In the same way as $2 \times 2 = 4 \therefore \sqrt[2]{4} = 2$ $= \sqrt[2]{2 \times 2} = 2^{1/2} \times 2^{1/2} = 2^{1/2+1/2}$
$= 2^1 = 2$

Then $a^{1/2} \times a^{1/2} = a^1 \therefore \sqrt[2]{a^1} = a^{1/2}$

Similarly $a^{1/3} \times a^{1/3} \times a^{1/3} = a^1 \therefore \sqrt[3]{a^1} = a^{1/3}$

And $a^{2/3} \times a^{2/3} \times a^{2/3} = a^2 \therefore \sqrt[3]{a^2} = a^{2/3}$

In general:

$$\boxed{a^{m/n} = \sqrt[n]{a^m}}$$

So to find the n^{th} root of a quantity we divide the index of the quantity m by n.

Examples

Example: $\sqrt[7]{a^3 b^4 x^7} = a^{3/7} \times b^{4/7} \times x^{7/7}$
$= a^{3/7} \times b^{4/7} \times x$

$$\sqrt[4]{a^2} = a^{2/4}$$

$$\sqrt[5]{y^4} = y^{4/5}$$

This rule enables us to evaluate expressions such as $81^{1/4}$, $16^{3/4}$ without the use of a calculator by changing the number part into a power and then applying rule (5), the power rule.

$$81^{1/4} = (3^4)^{1/4} = 3^{4 \times 1/4} = 3^1 = 3$$

$$16^{3/4} = (2^4)^{3/4} = 2^{4 \times 3/4} = 2^3 = 8$$

Further Examples

$$27^{1/3} = (3^3)^{1/3} = 3^{3 \times 1/3} = 3^1 = 3$$

$$(125)^{1/3} = (5^3)^{1/3} = 5^{3 \times 1/3} = 5^1 = 5$$

$$8^{-1/3} = \frac{1}{8^{1/3}} = \frac{1}{(2^3)^{1/3}} = \frac{1}{2^{3 \times 1/3}} = \frac{1}{2}$$

Summary

Rule 1	$a^m \times a^n = a^{m+n}$
Rule 2	$\dfrac{a^m}{a^n} = a^{m-n}$
Rule 3	$a^{-m} = \dfrac{1}{a^m}$
Rule 4	$a^0 = 1$
Rule 5	$(a^m)^n = a^{m \times n}$
Rule 6	$a^{m/n} = \sqrt[n]{a^m}$

Denary System

We are used to calculating in Base 10 (***Denary***). In this base, we use the 10 digits 0, 1, 2, 3, 4, 5,6, 7, 8 , 9. When adding or multiplying, once we get above 9, we have to write down the right hand digit and carry the left hand digit to the next column.

For example take the number 13, write down 3 and carry 1. This is because we have 3 units and 1 ten.

Since early school years, we have been familiar with the column heads Thousands, Hundreds, Tens and Units, under which we arrange the digits of large numbers. If we convert these headings into powers of 10 we get:

	Thousands		Hundreds		Tens		Units
	10^3		10^2		10^1		10^0
So 2368 =	2		3		6		8
ie.	2×10^3	+	3×10^2	+	6×10^1	+	8×10^0

In the Base 10 (denary) system, the highest value digit in any column is always one less than the base number. i.e. 10 - 1 = 9

So for other number bases, we would expect the same ruling.

Base 2 (binary) highest value digit is 2 - 1 =1. So in the binary system we use the two digits 0 and 1.

Base 8 (octal) highest value digit is 8 - 1 = 7. So in the octal system we use the digits 0, 1, 2, 3, 4, 5, 6, 7.

Base 16 (hexadecimal) highest value digit is 16 - 1 = 15.
So in the hexadecimal system we use the digits, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 and 15.