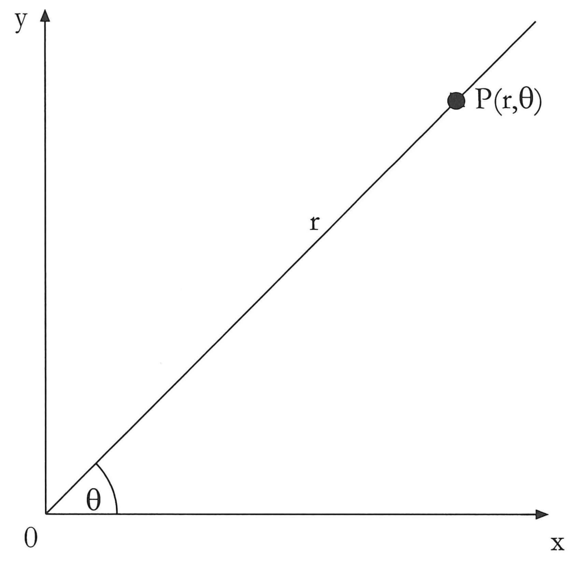
Polar Coordinates

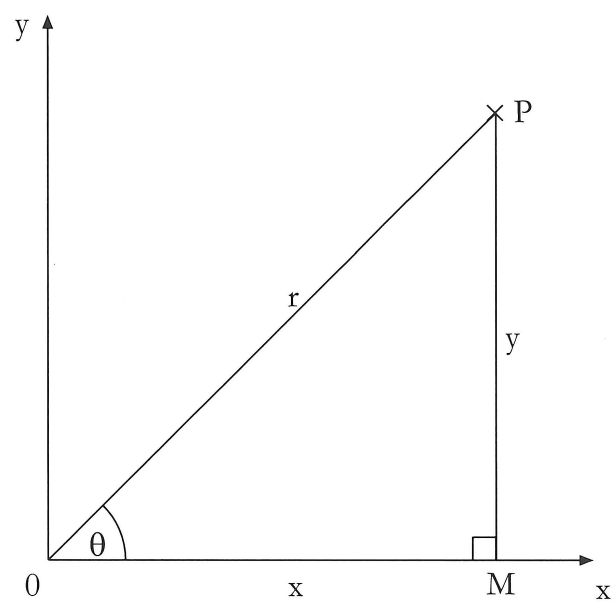
So far, we have used Cartesian coordinates (x, y) to give the position of a point in a plane relative to two axis. We will now look at an alternative system of coordinates in which the position of a point is described in terms of its distance and direction from a fixed point.



Let us suppose that 0x is a straight line with the point 0 fixed. If the distance 0P = r and the angle $\angle$x0P = θ, then the position of the point P is given by (r, θ). These are called the polar coordinates of the point P.

If the angle θ is measured in an anticlockwise direction from 0x it is deemed to be ***positive***, whilst those measured in a clockwise direction from 0x are ***negative***.

Relationship between Cartesian and Polar Coordinates



Consider the right angled triangle, POM. Using our knowledge of trigonometry we can see that:

$\frac{x}{r} = \cos\theta$ and $\frac{y}{r} = \text{Sin}\,\theta$

multiplying both sides by r

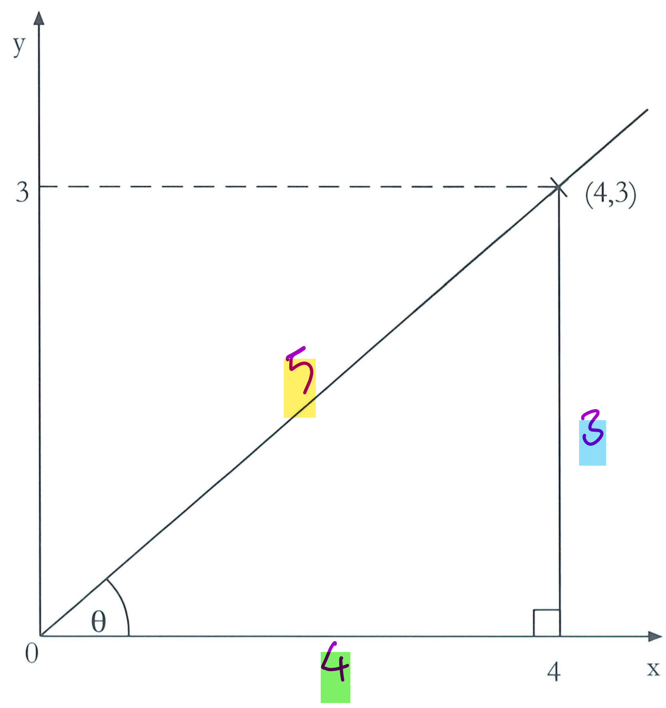
$x = r\,\text{Cos}\,\theta$ $\quad$ $y = r\,\text{Sin}\,\theta$

also $\frac{y}{x} = \text{Tan}\,\theta$

and $r = \sqrt{x^2 + y^2}$ (by Pythagoras' theorem)

By using the above relationships it is easy to convert from Cartesian to polar coordinates and vice versa.

Example 1 Find the polar coordinates of the point (4,3)



From the above: $\text{Tan}\,\theta = \frac{3}{4} = 0.75$

ie $\theta = 36.9°$

also $r = \sqrt{x^2 + y^2}$

$r = \sqrt{3^2 + 4^2}$

$r = \sqrt{25}$

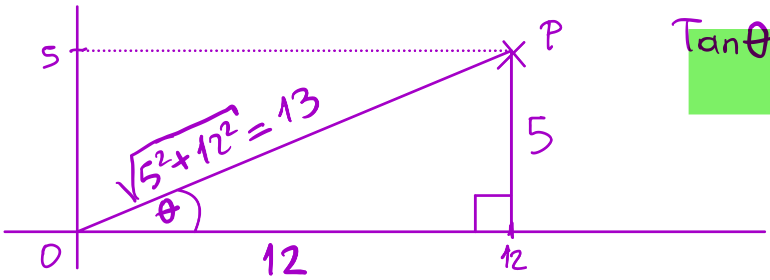
$r = 5$

The point (4,3) is (5, 36.9°) in polar form

$\text{Tan}^{-1}0.75 = \theta = 36.8699$
$\cong 36.9°$

$(r, \theta) = (5, 36.9°)$

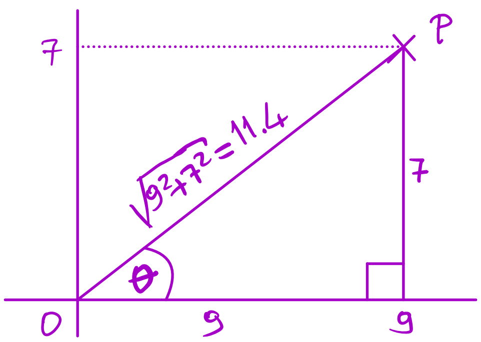
Example: Find the polar coordinates of (12,5)



$\tan\theta=\frac{5}{12} \Rightarrow \theta=\tan^{-1}\frac{5}{12}$

$\boxed{\theta=22.6199}$

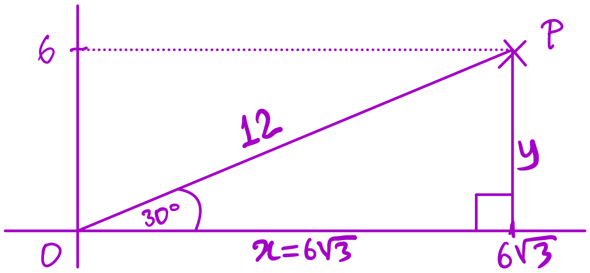
Example: Find the polar coordinates of (9,7)



$\tan\theta=\frac{7}{9} \Rightarrow \theta=\tan^{-1}\frac{7}{9}$

$\boxed{\theta=37.8750}$

Example: Express in Cartesian form the point $(12, 30°)$



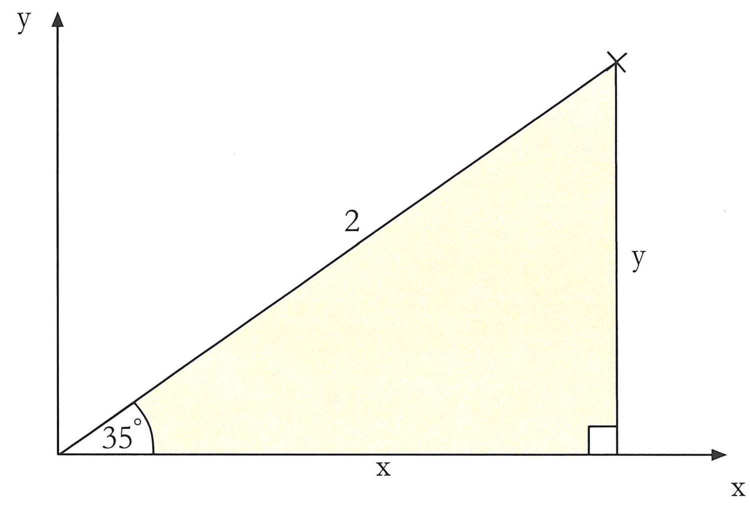
$x=r\cos 30° = 12*\frac{\sqrt{3}}{2}=6\sqrt{3}$

$y=r\sin 30° = 12*\frac{1}{2}=6$

Cartesian form of $(12,30°)$ is $(6\sqrt{3}, 6)$

Example 2

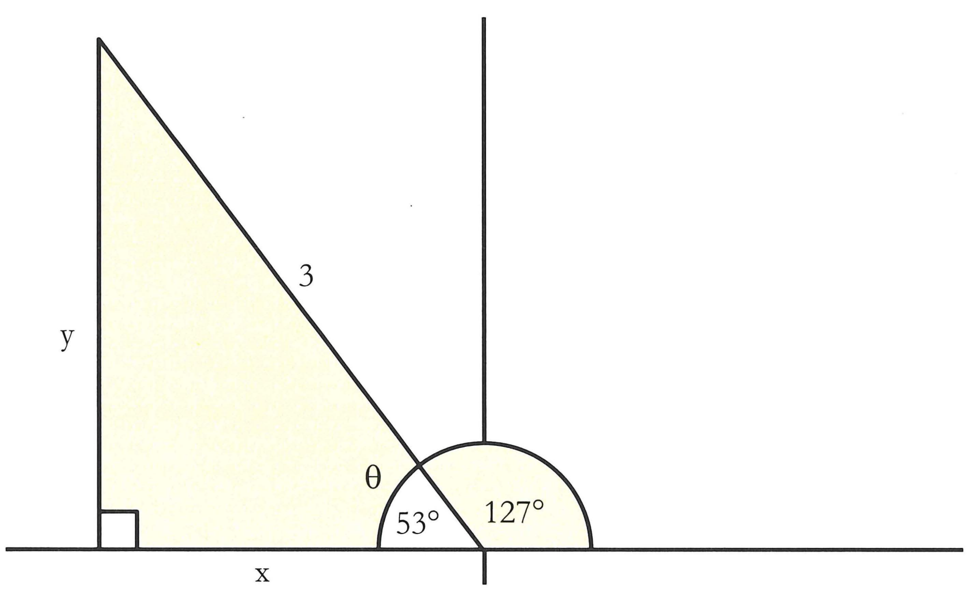
Express in Cartesian form the point (2,35°)



$\frac{x}{2}$ = Cos 35°	$\frac{y}{2}$ = Sin 35°
x = 2 x Cos 35°	y = 2 x Sin 35°
x = 1.64 (2dp)	y = 1.15 (2dp)

Since the x and y coordinates are positive the Cartesian form of (2,35°) is (1.64, 1.15)

Example 3 Express in Cartesian form the point (3,127°)



From the diagram we can see that the x coordinate is negative.

$\theta = 180° - 127° = 53°$

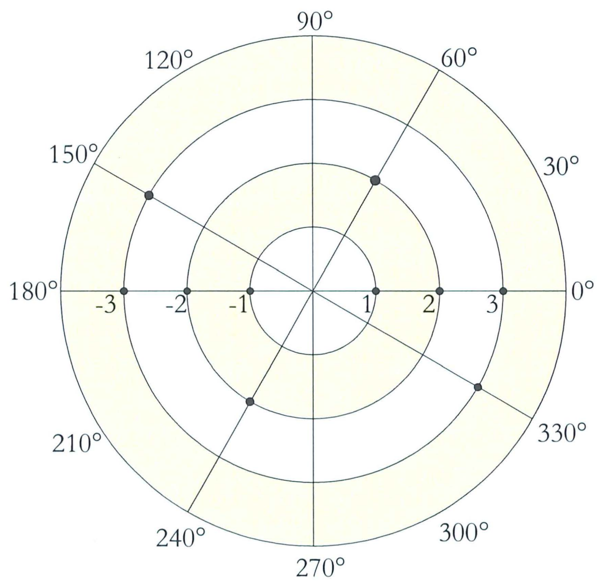
$\frac{x}{3} = \text{Cos } 53°$ $\qquad$ $\frac{y}{3} = \text{Sin } 53°$

$x = 3 \times \text{Cos } 53°$ $\qquad$ $y = 3 \times \text{Sin } 53°$

$x = -1.81$ $\qquad$ $y = 2.40$

$\therefore$ Cartesian form of (3,127°) is (–1.81, 2.40)

A negative value of r means the radius length is extended back through 0 in the opposite direction from the normal angle position.



In polar coordinates, you can define a point in two ways. For example, (2,60°) could be also defined as (–2,240°) and (3,330°)as (–3, 150°)

Example

Find the Cartesian coordinates for the point (– 5,130). i.e. This could be defined as (5,310°)

So: $\frac{x}{5} = \text{Cos } 310°$

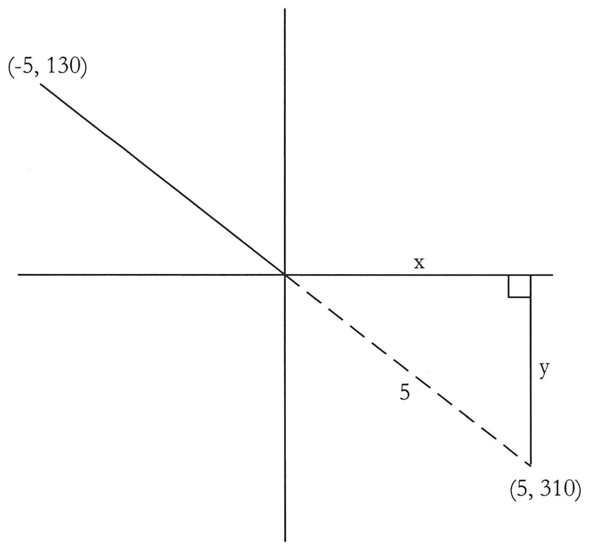
$x = 5 \times \text{Cos } 310°$

$x = 3.21$

$\frac{y}{5} = \text{Sin } 310°$

$y = 5 \times \text{Sin } 310°$

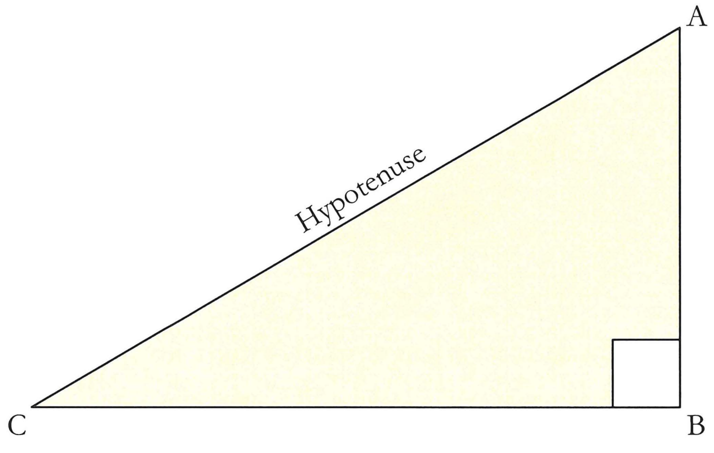
$y = -3.83$



i.e. The Cartesian coordinates are (3.21, - 3.83)

Pythagoras' Theorem

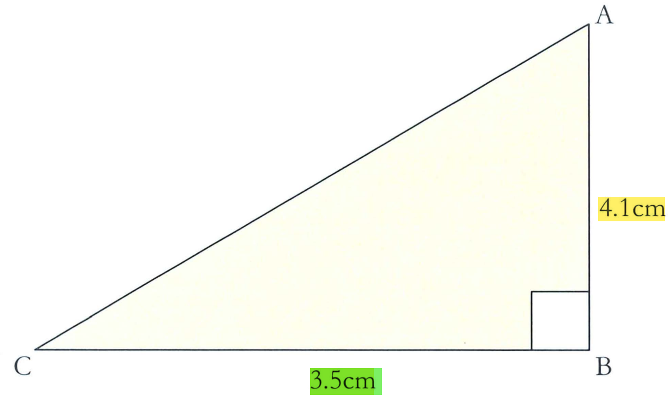
This states, "***In a right angled triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides***".



i.e. $AC^2 = AB^2 + BC^2$

In any right-angled triangle, the hypotenuse is the longest side and always lies opposite to the right angle.

Example 1 Find the hypotenuse AC in the following diagram:



By Pythagoras' theorem:

$AC^2 = AB^2 + BC^2$

$AC^2 = 4.1^2 + 3.5^2$

$AC^2 = 16.81 + 12.25$

$AC^2 = 29.06$

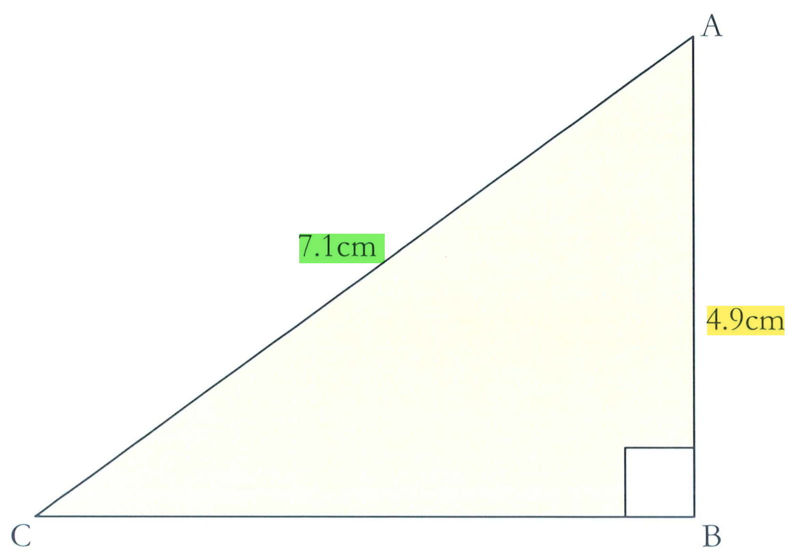
Taking square roots of both sides.

$AC = \sqrt{29.06}$

$AC = 5.39\text{cm (2dp)}$

Example 2

Find the side BC in the following triangle:



From Pythagoras' theorem: $AC^2 = AB^2 + BC^2$

Since we want to find BC, we can rearrange the equation to make BC^2 the subject:

Subtract AB^2 from both sides

$BC^2 = AC^2 - AB^2$

$BC^2 = 7.1^2 - 4.9^2$

$BC^2 = 50.41 - 24.01$

$BC^2 = 26.4$

$BC = \sqrt{26.4}$

$BC = 5.14$cm (2dp)

It is worth noting that there are certain standard right-angled triangles. For example, triangles with sides of 3 : 4 : 5, 5 : 12 : 13, 7 : 24 : 25 and multiples of them are right-angled triangles.

Summary

To find the hypotenuse given the other two sides then:

a) Square each of the given sides

b) Add the results.

c) Find square root of answer.

To find one of the other sides given the hypotenuse and the third side then:

a) Square the hypotenuse and square the given side.

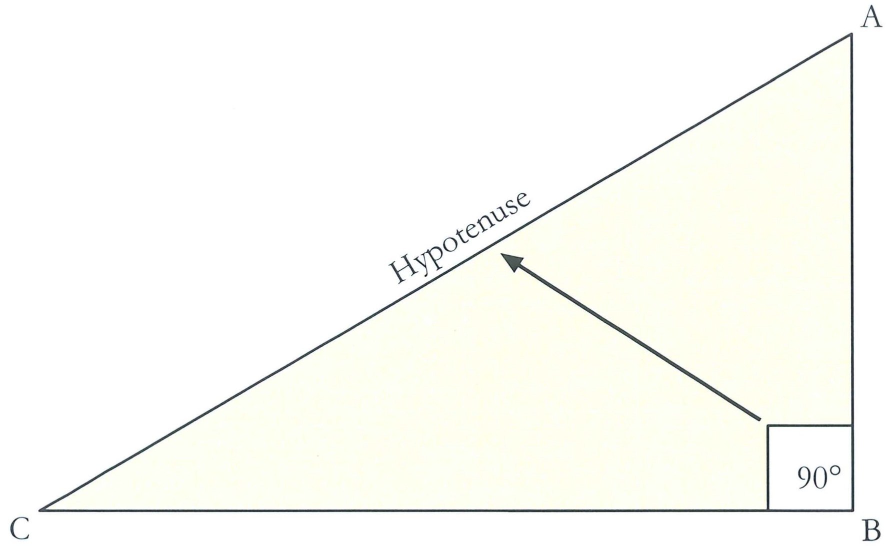
b) Subtract the square of the given side from the square of the hypotenuse.

c) Find the square root of the answer.

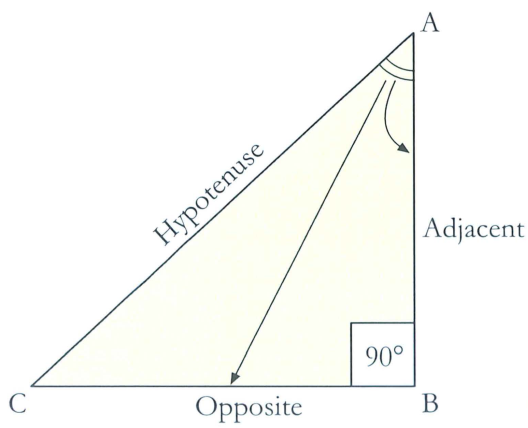
Trigonometry

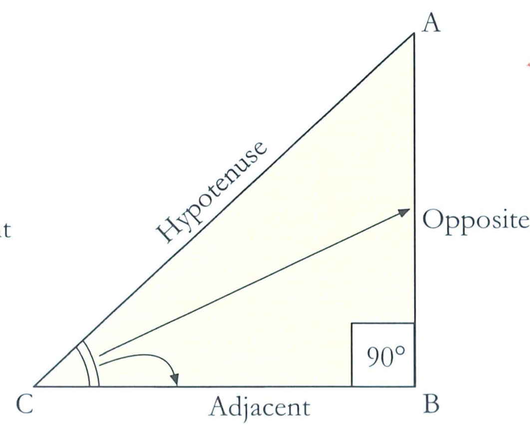
A right-angled triangle is one in which one of the angles is equal to 90°.

Since this angle is the largest in the triangle (the other two only add up to 90° between them), then the side opposite to it is the longest side. It is called the ***hypotenuse***.

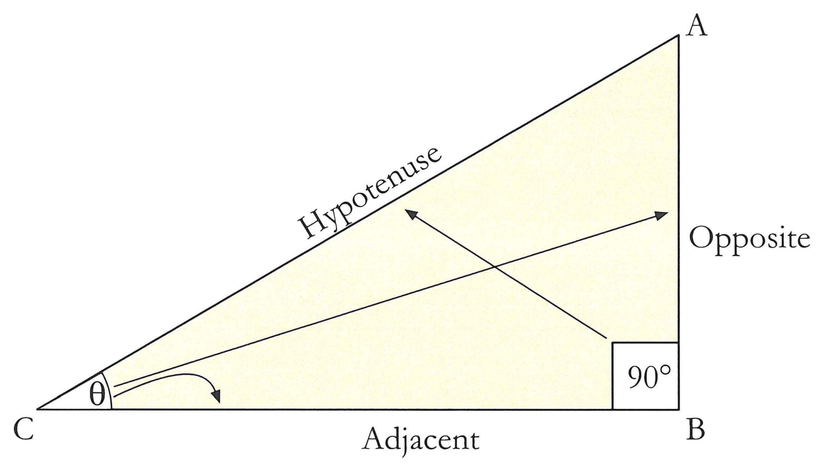


If we now consider the angle C, then the side opposite to it AB is called the ***opposite***. The side BC is called the ***adjacent***. However, if we now consider the angle A, then the side BC is now the ***opposite*** and the side AB is the ***adjacent***. So, although the hypotenuse ***is always opposite*** to the right angle, the other two sides (opposite and adjacent) depend upon which of the two other angles A and C we are looking at.





If we now consider the following right angled triangle:



The ratio $\frac{AB}{AC}$ ie $\frac{\text{opposite}}{\text{hypotenuse}}$ is called the Sine of angle θ

The ratio $\frac{BC}{AC}$ ie $\frac{\text{adjacent}}{\text{hypotenuse}}$ is called the Cosine of angle θ

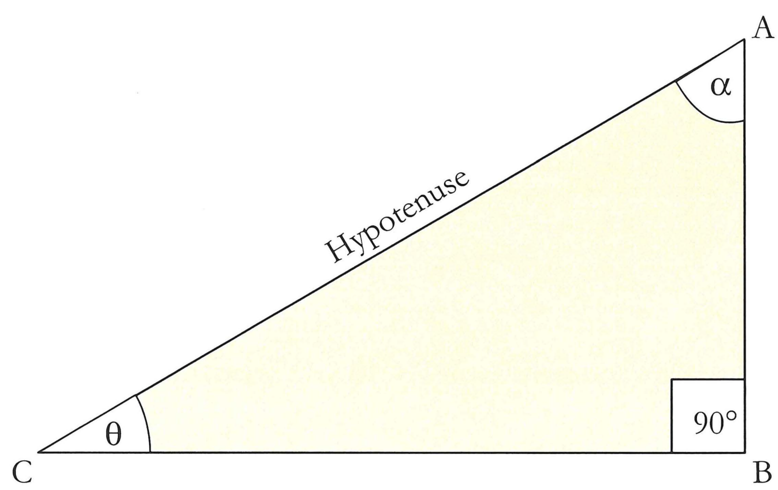
The ratio $\frac{AB}{BC}$ ie $\frac{\text{opposite}}{\text{adjacent}}$ is called the Tangent of angle θ

These ratios can be used to solve a right-angled triangle if we are given:

- One angle (other than the right angle) and a side.
- Two of the three sides.

Let us now look at these three ratios in turn.

Sine of an Angle (Abbreviation: Sin)



From above: $\text{Sin } \theta = \dfrac{AB}{AC} = \dfrac{\text{opposite}}{\text{hypotenuse}}$

On the other hand: $\text{Sin } \alpha = \dfrac{BC}{AC} = \dfrac{\text{opposite}}{\text{hypotenuse}}$

The sine, cosine or tangent of an angle is obtained by using tables or a scientific calculator.

Although calculators are not allowed in the examination the following is described for your information and guidance only.

Example

To find the Sine of 56° using a scientific calculator, the input is: -

Sin	5	6	=

The answer is 0.8390

To find the tangent of 30°, the input is:

Tan	3	0	=

The answer is 0.5774

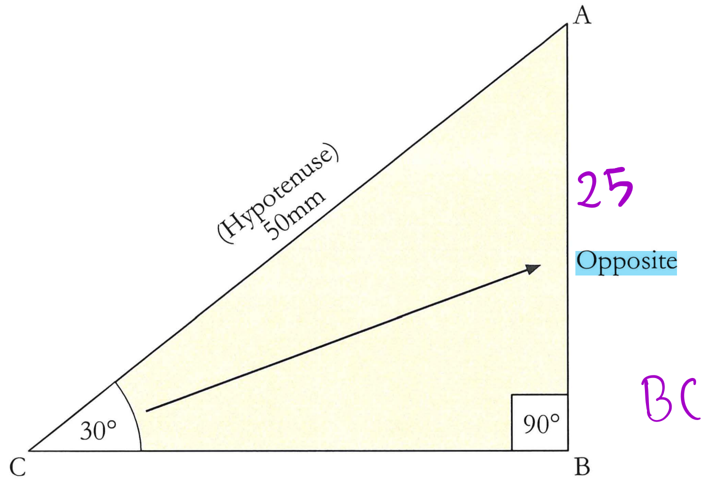
Note: This procedure is reversed for some calculators. You may have to input the degrees before pressing the Sin/Cos/Tan button.

To find the cosine of 40°, the input would be:

4	0	Cos	=

The answer is 0.7660

If we now look at a complete example:



$BC = \sqrt{50^2 - 25^2} = 43.3$

Find the length of AB in the triangle

AB is opposite to the given angle 30°

AC is the given side (***hypoteneuse***)

$$\sin 30° = \frac{AB}{AC}$$

$$\sin 30° = \frac{AB}{50}$$

Multiplying both sides by 50

$$\sin 30° \times 50 = \frac{AB}{\cancel{50}} \times \cancel{50}$$

which gives:

AB = Sin 30° x 50

AB = 0.5 x 50

AB = 25mm

$\cos 30° = \frac{BC}{AC}$

$BC = AC * \cos 30°$

$BC = 50 * \cos 30° =$

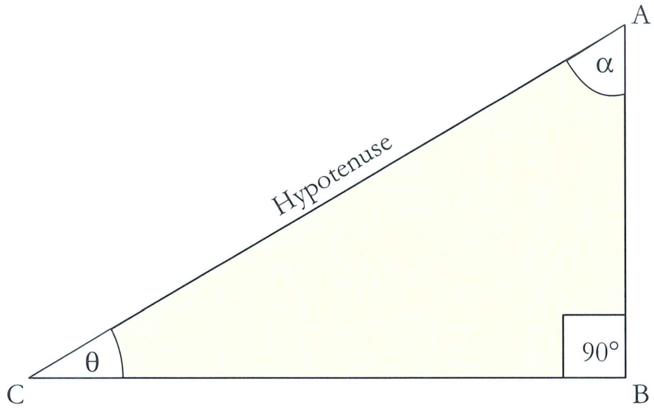
$\boxed{BC = 43.3}$

If a scientific calculator is used for this then the input is: -

Sin	3	0	x	5	0	=

The answer is 25 mm

Cosine of an Angle

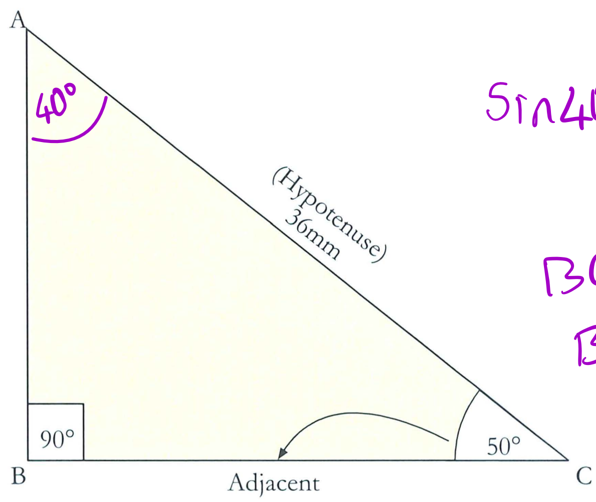


Now Cos θ = $\frac{BC}{AC}$ = $\frac{\text{adjacent}}{\text{hypotenuse}}$ and Cos α = $\frac{AB}{AC}$ = $\frac{\text{adjacent}}{\text{hypotenuse}}$

$\sin\alpha = \frac{BC}{AC}$ $\sin\theta = \frac{AB}{AC}$ so $\cos\theta = \sin\alpha$

$\cos\alpha = \sin\theta$

Example Find the length of the side BC in the triangle.



$\sin 40^\circ = \frac{BC}{AC} = \frac{BC}{36}$

$BC = 36 * \sin 40^\circ$

$BC = 23.1$ mm

BC is adjacent to the given angle 50°

AC is the given side (hypotenuse)

$\frac{AB}{AC} = \sin 50^\circ = \cos 40^\circ$

$AB = AC * \sin 50^\circ$

$AB = AC * \cos 40$

$\boxed{AB = 27.5776}$

Cos 50° = $\frac{BC}{AC}$

Cos 50° = $\frac{BC}{36}$

Multiplying both sides by 36

Cos 50° x 36 = $\frac{BC}{\cancel{36}}$ x $\cancel{36}$

BC = Cos 50° x 36

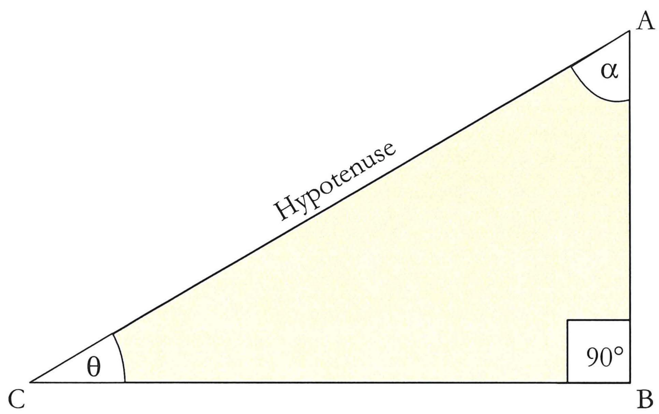
BC = 0.6428 x 36

BC = 23.1mm (1 dp)

The input on the calculator is:

Cos	5	0	x	3	6	=

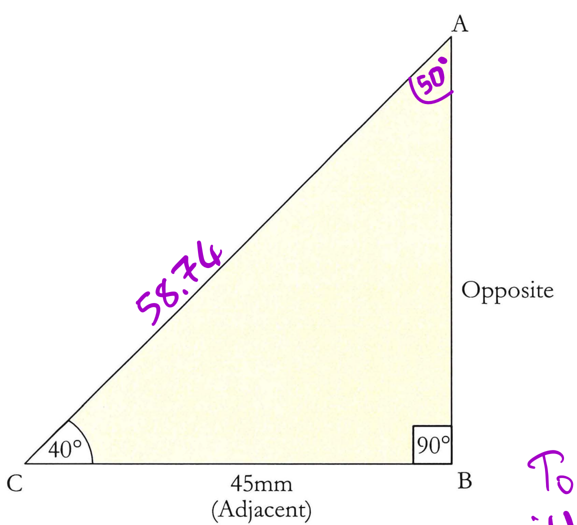
Tangent of an Angle



Now Tan $\theta = \frac{AB}{BC} = \frac{\text{opposite}}{\text{adjacent}}$ and Tan $\alpha = \frac{BC}{AB} = \frac{\text{opposite}}{\text{adjacent}}$

Example 1

Find the length of the side AB in the triangle



$\text{Tan } 40° = \frac{AB}{BC}$

$\text{Tan } 40° = \frac{AB}{45}$

Multiplying both sides by 45

$\text{Tan } 40° \times 45 = \frac{AB}{\cancel{45}_1} \times \cancel{45}^1$

AB = Tan 40° x 45

AB = 0.8391 x 45

AB = 37.8mm (1 dp)

To determine the hypotenuse use either cosinus or sinus function

$\text{Cos}40° = \frac{45}{AC} \Rightarrow AC = \frac{45}{\text{Cos}40°}$

The input on the calculator is:

Tan	4	0	x	4	5	=

$\text{Sin}50° = \frac{BC}{AC}$

$\boxed{AC = 58.74}$

$AC = \frac{BC}{\text{Sin}50} = \frac{45}{\text{Sin}50}$

In all of the examples so far, the unknown side (the one we want to find) has always been on the top of the ratio.

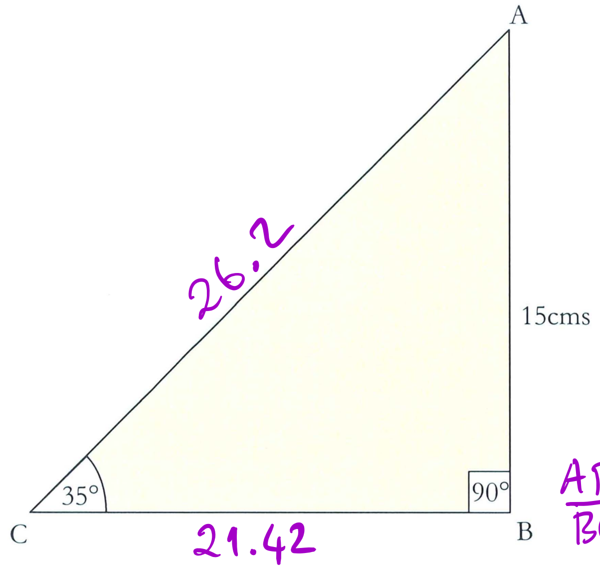
Consider the following however:

Example 2

Find AC

AC is the hypotenuse

AB is the given side (opposite)



$$\text{Sin } 35° = \frac{AB}{AC}$$

$$\text{Sin } 35° = \frac{15}{AC}$$

Multiplying both sides by AC

$$AC \times \text{Sin } 35° = \frac{15}{\cancel{AC}_1} \times \cancel{AC}^1$$

$$AC \times \text{Sin } 35° = 15$$

Dividing both sides by Sin 35°

$$\frac{AC \times \cancel{\text{Sin } 35°}^1}{\cancel{\text{Sin } 35°}_1} = \frac{15}{\text{Sin } 35°}$$

$$AC = \frac{15}{0.5736}$$

$$AC = 26.2\text{cm (1 dp)}$$

$$\frac{AB}{BC} = \tan 35°, \quad BC = \frac{AB}{\tan 35°}$$

$$BC = \frac{15}{\tan 35°}$$

$$\boxed{BC = 21.42}$$

The input on the calculator is:

1	5	÷	Sin	3	5	=

Example 3

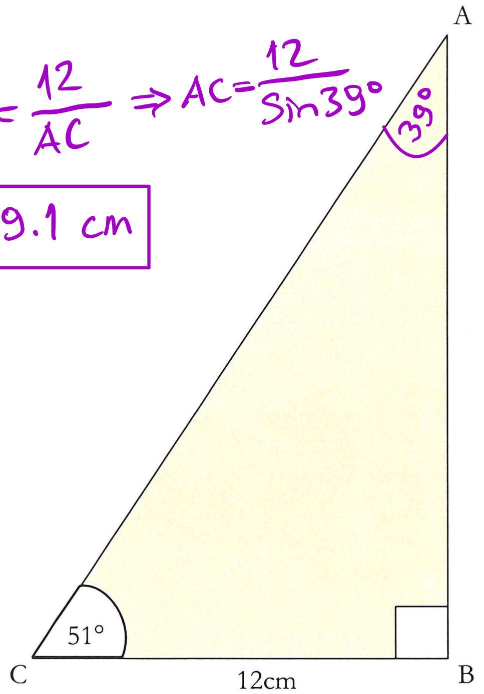
Find the side AC in the following triangle

AC is the hypotenuse

BC is the given side (adjacent)

$\sin 39^\circ = \frac{12}{AC} \Rightarrow AC = \frac{12}{\sin 39^\circ}$

$\boxed{AC = 19.1 \text{ cm}}$



$$\cos 51^\circ = \frac{BC}{AC}$$

$$\cos 51^\circ = \frac{12}{AC}$$

Multiplying both sides by AC

$$AC \times \cos 51^\circ = \frac{12}{\cancel{AC}^{1}} \times \cancel{AC}^{1}$$

$$AC \times \cos 51^\circ = 12$$

Dividing both sides by Cos 51°

$$AC = \frac{12}{\cos 51^\circ}$$

$$AC = \frac{12}{0.6293}$$

AC = 19.1cm (1 dp)

$\frac{AB}{BC} = \tan 51$

$AB = BC * \tan 51$

$AB = 12 * \tan 51$

$\boxed{AB = 14.8188}$

The input on the calculator is:

1	2	÷	Cos	5	1	=

$\tan 39^\circ = \frac{BC}{AB} \Rightarrow AB = \frac{BC}{\tan 39^\circ} = \frac{12}{\tan 39^\circ} = 14.8188$

Note: The tangent cannot be used in any of the above two examples, because the hypotenuse is the side to be found and it does not appear in the tangent ratio of:

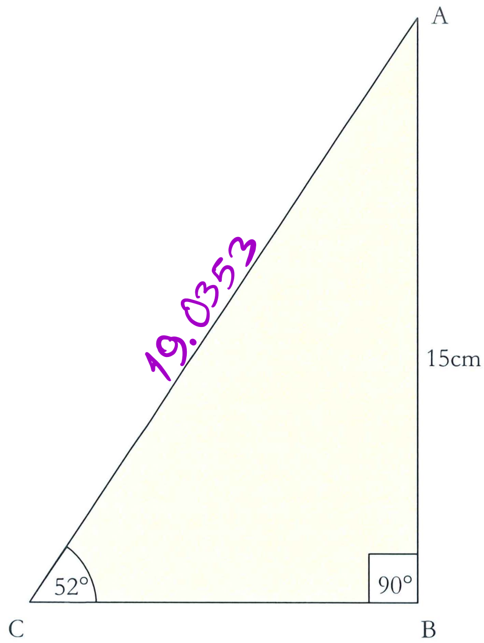
$$\frac{\text{opposite}}{\text{adjacent}}$$

However, consider the following example:

Find the side BC

BC is adjacent to the 52° angle

AB is the given side i.e. opposite



$$\text{Tan } 52° = \frac{AB}{BC}$$

$$\text{Tan } 52° = \frac{15}{BC}$$

Multiplying both sides by BC

$$\text{Tan } 52° \times BC = \frac{15}{\cancel{BC}_1} \times \cancel{BC}^1$$

$$BC = \frac{15}{\text{Tan } 52°}$$

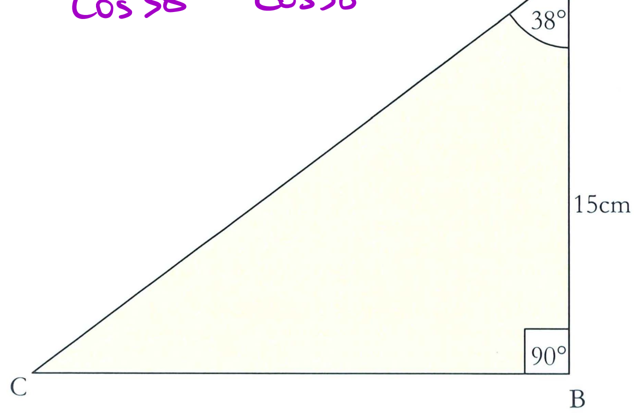
$$BC = 11.7\text{cm (1 dp)}$$

$$\frac{AB}{AC} = \text{Sin } 52°$$

$$AC = \frac{AB}{\text{Sin}52°} = \frac{15}{\text{Sin}52°} = 19.0353$$

If we change to angle A which is 90° - 52° = 38°

$$\frac{AB}{AC} = \text{Cos}38°, \quad AC = \frac{AB}{\text{Cos}38°} = \frac{15}{\text{Cos}38} = 19.0353$$



BC is opposite to 38° AB is the given side i.e. adjacent

$$\text{Tan } 38° = \frac{BC}{15}$$

Multiplying both sides by 15

$$\text{Tan } 38° \times 15 = \frac{BC}{\cancel{15}_1} \times \cancel{15}^1$$

$$BC = \text{Tan } 38° \times 15$$

$$BC = 0.7813 \times 15$$

$$BC = 11.7\text{cm (1 dp)}$$

So the same result is obtained by the slightly easier method.

Summary

If the unknown side appears on the ***top*** of the original ratio then: -

Unknown Side = Given side × Sin/Cos/Tan of the given angle

For example:

$$\text{if } \sin 30° = \frac{AB}{15}$$

$$\text{then } AB = 15 \times \sin 30°$$

If the unknown side appears on the ***bottom*** of the original ratio then:

Unknown Side = Given side ÷ Sin/Cos/or Tan of the given angle

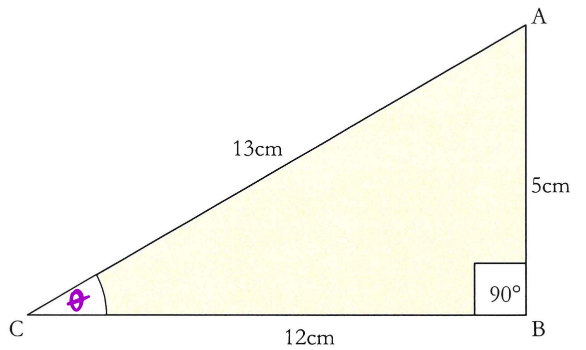
For example:

$$\text{if } \sin 50° = \frac{15}{AB}$$

$$\text{then } AB = \frac{15}{\sin 50°}$$

If a side other than the hypotenuse is required then it might be better to use the third angle in the triangle.

To Find an Angle Given Two or More Sides



Since the three sides are given angle C can be found by using any one of the three ratios.

$$\tan\theta = \frac{5}{12} \Rightarrow \theta = \tan^{-1}\frac{5}{12} = 22.6199$$

$$\sin\theta = \frac{5}{13} \Rightarrow \theta = \sin^{-1}\frac{5}{13} = 22.6199$$

$$\cos\theta = \frac{12}{13} \Rightarrow \theta = \cos^{-1}\frac{12}{13} = 22.6199$$

$$\text{Sin } \angle C = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{5}{13}$$

$$\text{i.e. Sin C} = \frac{5}{13}$$

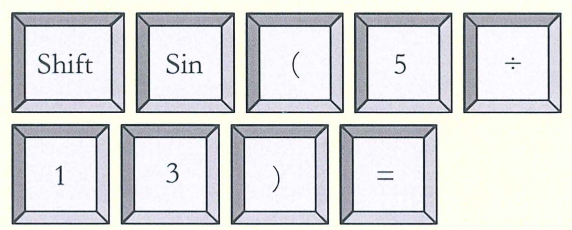
(But we don't want to find Sin C, we want to find $\angle C$)

$$\text{So } \; C = \text{Sin}^{-1} \frac{5}{13}$$

(ie Sin^{-1} by shift/sin on the calculator)

$$\angle C = 22.6° \text{ (1 dp)}$$

On the calculator:



In a similar way:

$$\text{Cos } \angle C = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{12}{13}$$

$$\angle C = \text{Cos}^{-1} \frac{12}{13}$$

$$C = 22.6° \text{ (1 dp)}$$

$$\text{and Tan } \angle C = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{5}{12}$$

$$\angle C = \text{Tan}^{-1} \frac{5}{12}$$

$$C = 22.6° \text{ (1 dp)}$$

Method to Choose Correct Trigonometrical Ratio

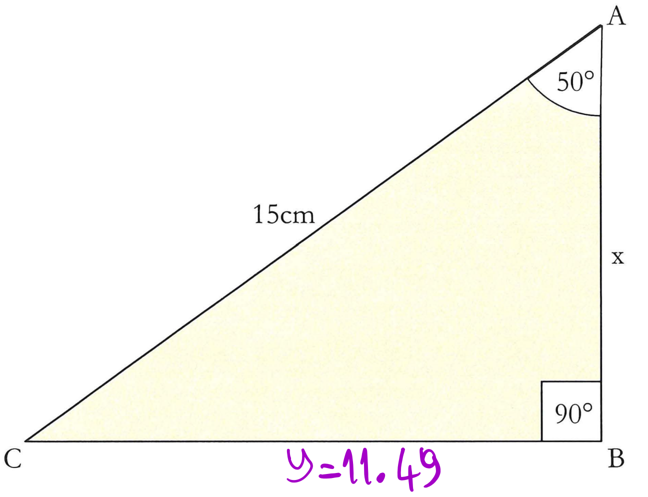
One of the questions many students ask when confronted with a trigonometrical problem is – "I know how to calculate but which ratio do 1 choose?"

A very simple method is: -

- Pinpoint where the unknown side is in relation to the given angle.
- Pinpoint where the given side is in relation to the given angle.
- Ask yourself "which of the three sides is ***not*** involved.
- Write down the three trig. Ratios and cross out the two that contain this un-required side.

Example

Find AB



y=15*Sin50°
y=11.49

Clearly AB is the required side. AC is the given side so BC is ***not*** required. In this problem BC is the opposite side: -

Cross out ratios with opposite:

$$\text{Sin} = \frac{\cancel{\text{Opposite}}}{\cancel{\text{Hypotenuse}}} \quad \text{Cos} = \frac{\text{Adjacent}}{\text{Hypotenuse}} \quad \text{Tan} = \frac{\cancel{\text{Opposite}}}{\cancel{\text{Adjacent}}}$$

Cos is the required ratio

$$\text{Cos } 50° = \frac{x}{15}$$

$$x = \text{Cos } 50 \times 15$$

$$x = 9.6\text{cm (1 dp)}$$

Types of Triangle

A polygon is a plane figure bounded by straight lines. Therefore a triangle (3 sides) is the smallest polygon possible. The sum of the angles of a triangle is 180^0 irrespective of its shape or size.

In any triangle, the longest side always lies opposite to the largest angle and the shortest side always lies opposite to the smallest angle. It follows that if two sides of a triangle are equal, then the angles opposite those sides must also be equal.

a) Acute Angled Triangle

As the name suggests an acute angled triangle has each of its three angles less than 90°.