INDUSTRIAL AUTOMATION & ROBOTICS TECHNOLOGY

Fundamentals + Transformation

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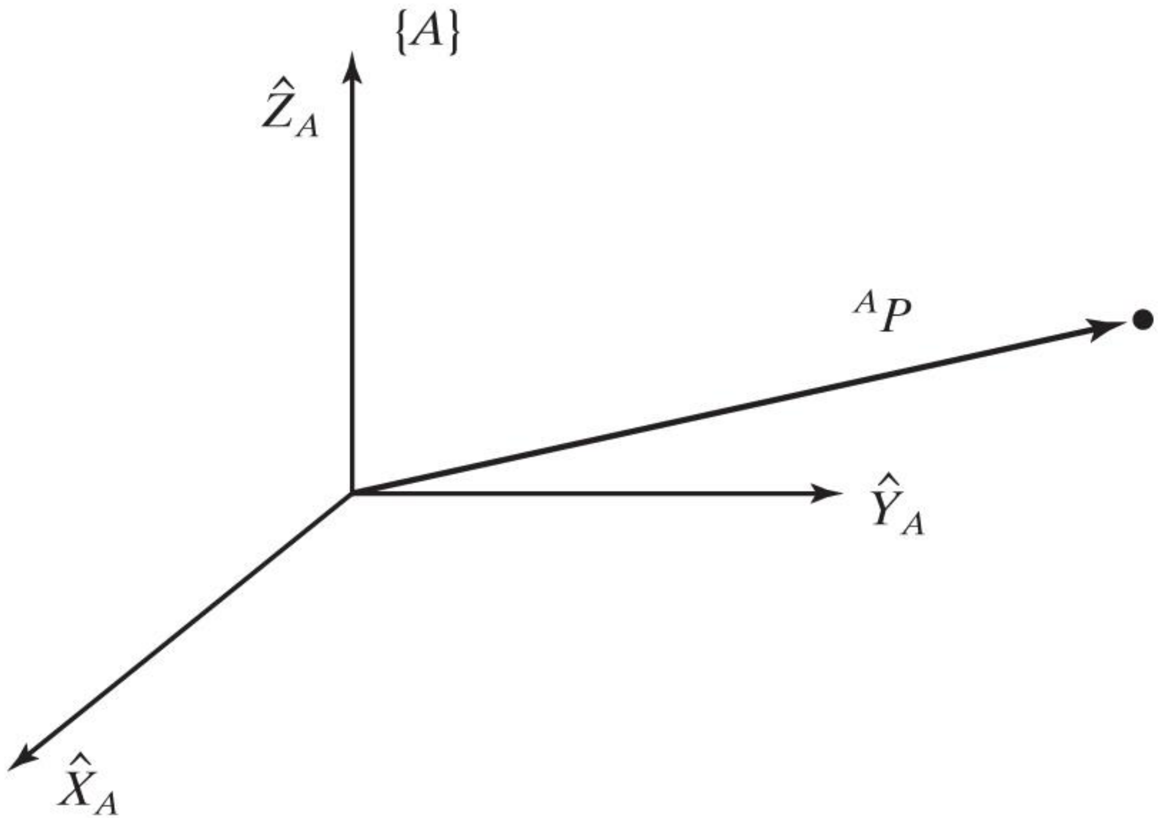
Materials used

- Chapter 2, Introduction to Robotics, John J. Craig
- Chapter 2, Introduction to Robotics, Saeed B. Niku

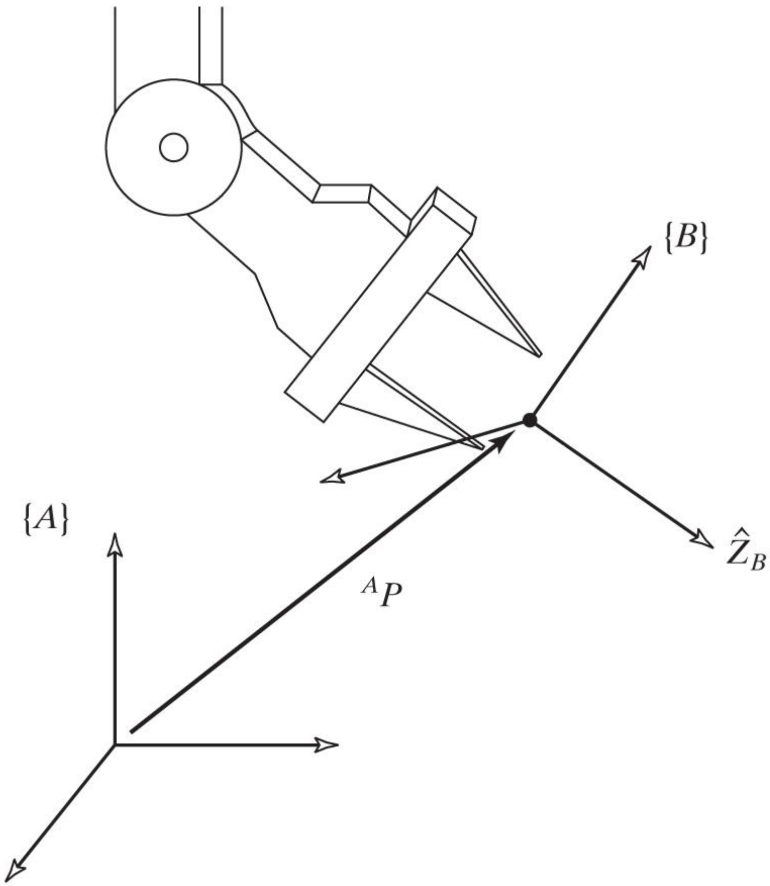
Position and Orientation

$$ {}^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} $$



Position and Orientation

- Coordinate system {B}
- Coordinate system {A}



Robot Kinematics

Forward Kinematics:

to determine where the robot's hand is?

(If all joint variables are known)

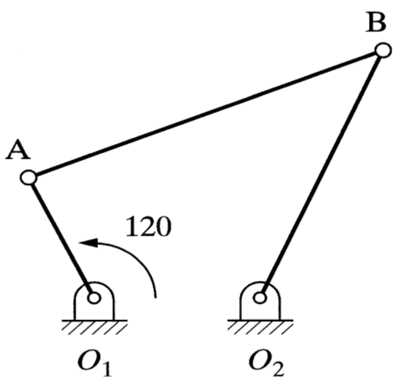
Inverse Kinematics:

to calculate what each joint variable is?

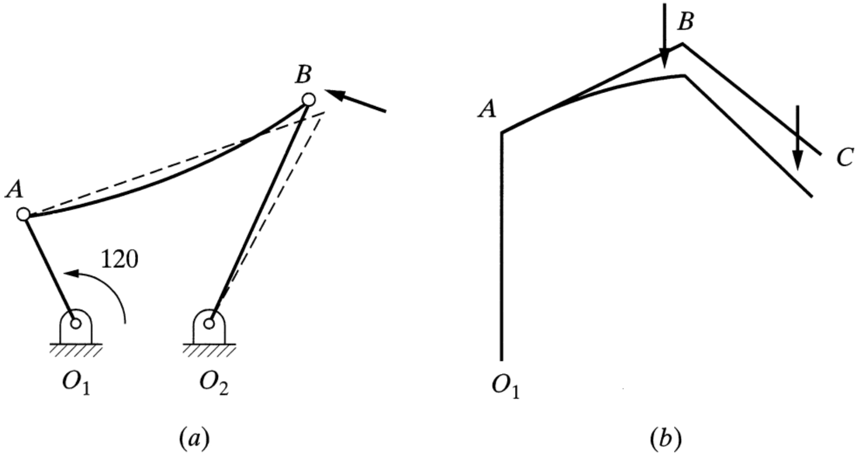
(If we desire that the hand be located at a particular point)

Robots as Mechanisms

Multiple type robot have multiple DOF.
(3 Dimensional, open loop, chain mechanisms)



A one-degree-of-freedom closed-loop four-bar mechanism



(a) Closed-loop versus (b) open-loop mechanism

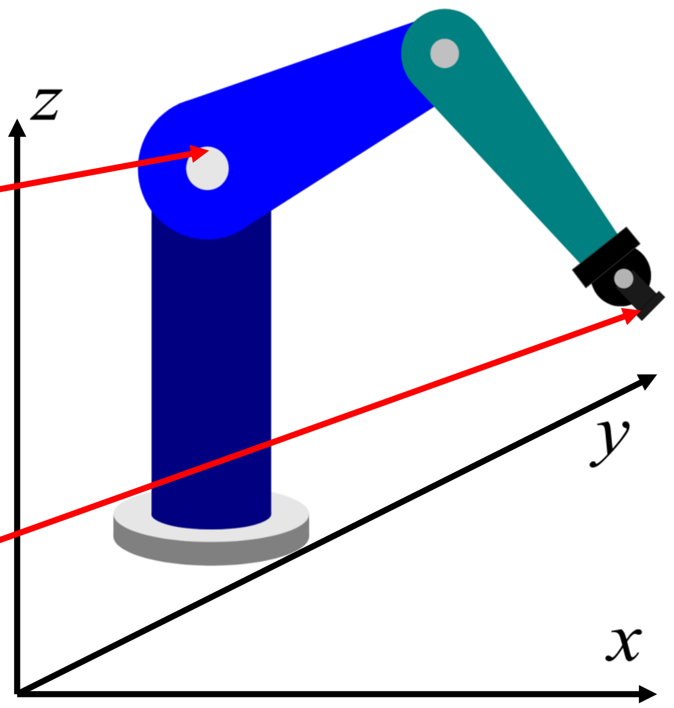
What is Kinematics

- Forward kinematics

Given joint variables

q

{p,R}



End-effector position and orientation, -Formula?

What is Kinematics

- Inverse kinematics

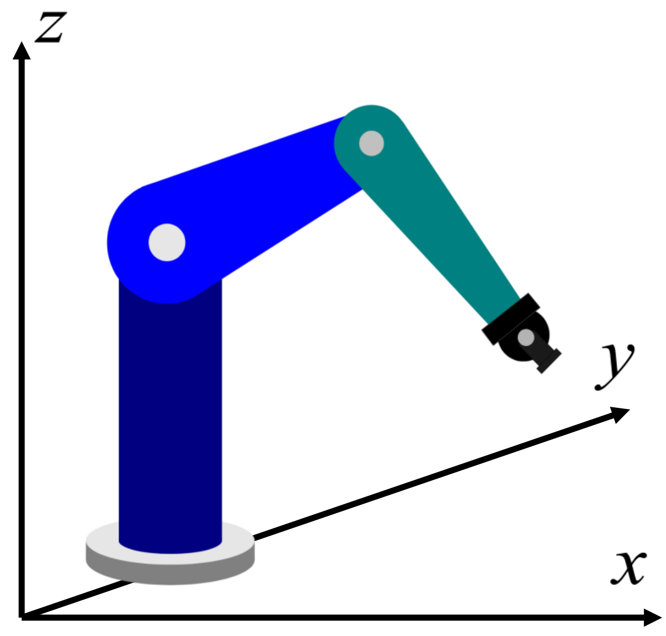
End effector position

and orientation

{p,R}

q

Joint variables -Formula?



Example

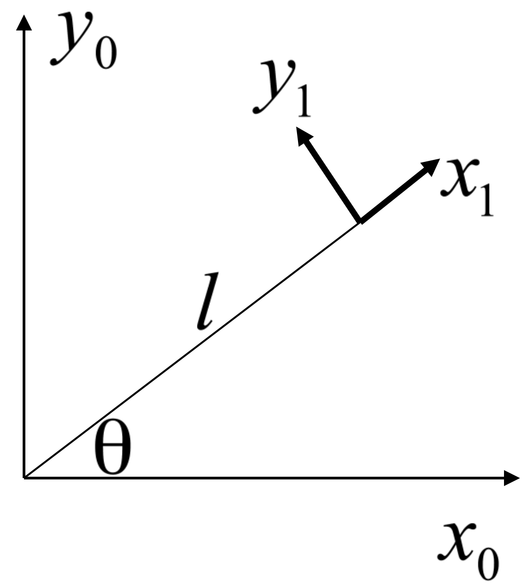
Forward kinematics

$$x_0 = l\cos\theta$$

$$y_0 = l\sin\theta$$

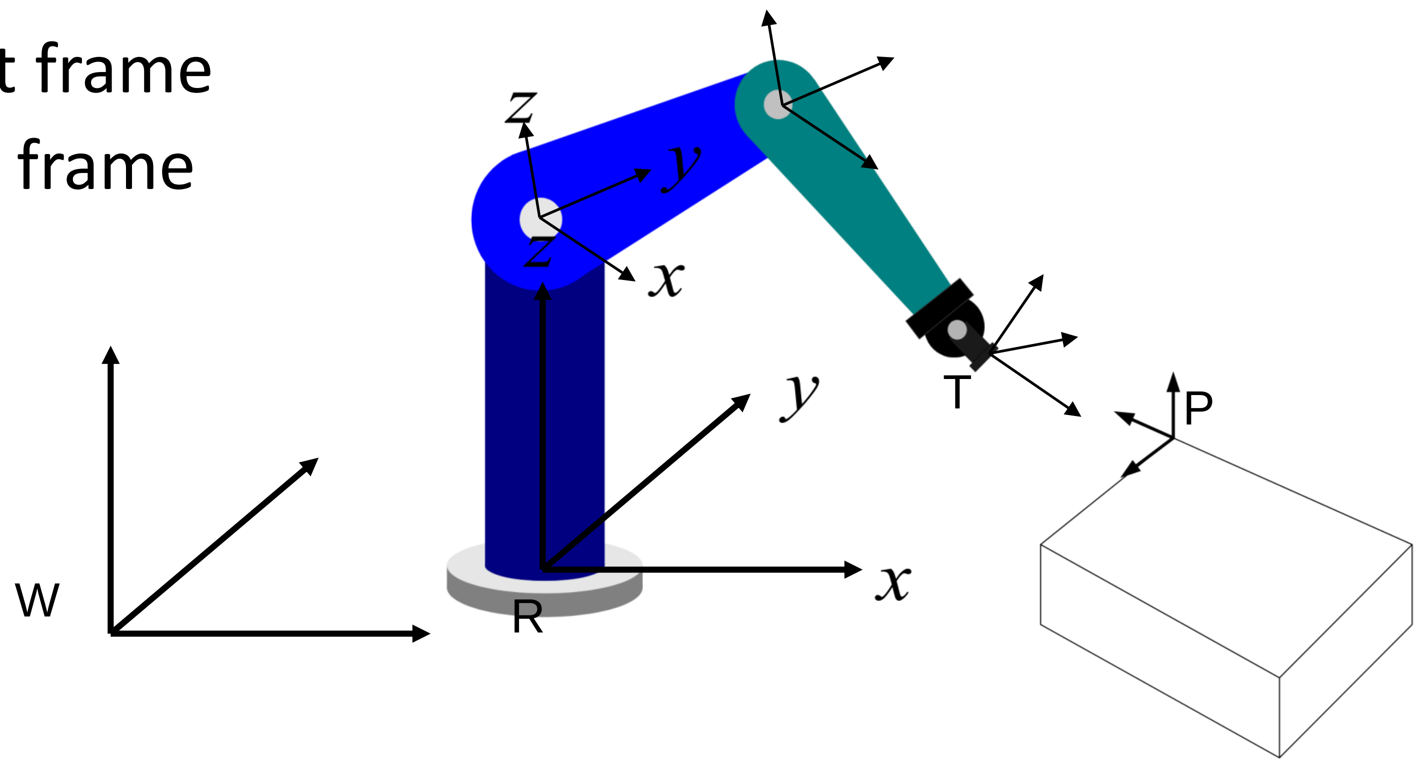
Inverse kinematics

$$\theta = \cos^{-1}(x_0 / l)$$



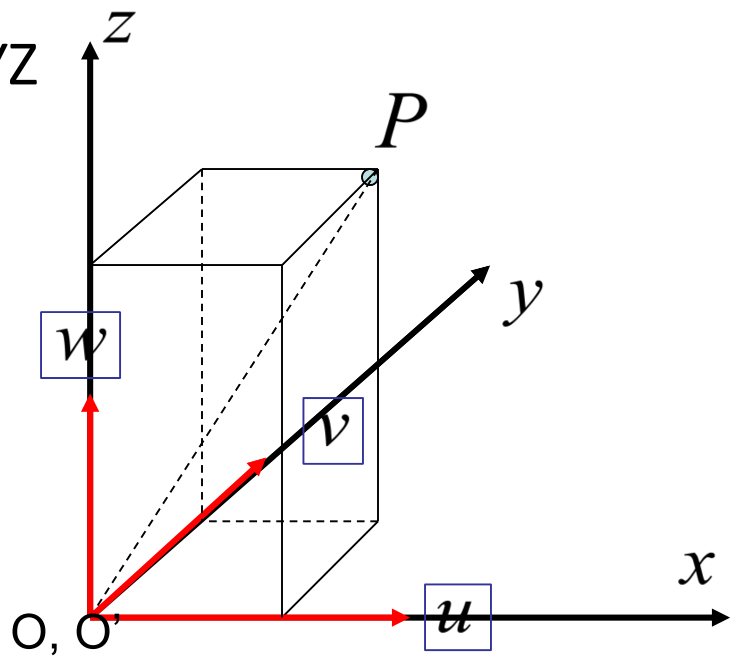
Preliminary

- Robot Reference Frames
  - World frame
  - Joint frame
  - Tool frame



Preliminary

- Coordinate Transformation
  - Reference coordinate frame OXYZ
  - Body-attached frame O'uvw



Point represented in OXYZ:

$$P_{xyz} = [p_x, p_y, p_z]^T$$

$$P_{xyz} = p_x \mathrm{i}_\mathrm{x} + p_y \mathrm{j}_\mathrm{y} + p_z \mathrm{k}_\mathrm{z}$$

Point represented in O'uvw:

$$P_{uvw} = p_u \mathrm{i}_\mathrm{u} + p_v \mathrm{j}_\mathrm{v} + p_w \mathrm{k}_\mathrm{w}$$

Two frames coincide ==> $p_u = p_x \quad p_v = p_y \quad p_w = p_z$

Preliminary

Properties: Dot Product

Let x and y be arbitrary vectors in R^3 and θ be the angle from x to y, then

$$x \cdot y = |x||y|\cos\theta$$

Properties of orthonormal coordinate frame

- Mutually perpendicular

$$i \cdot j = 0$$
$$i \cdot k = 0$$
$$k \cdot j = 0$$

- Unit vectors

$$|i| = 1$$
$$|j| = 1$$
$$|k| = 1$$

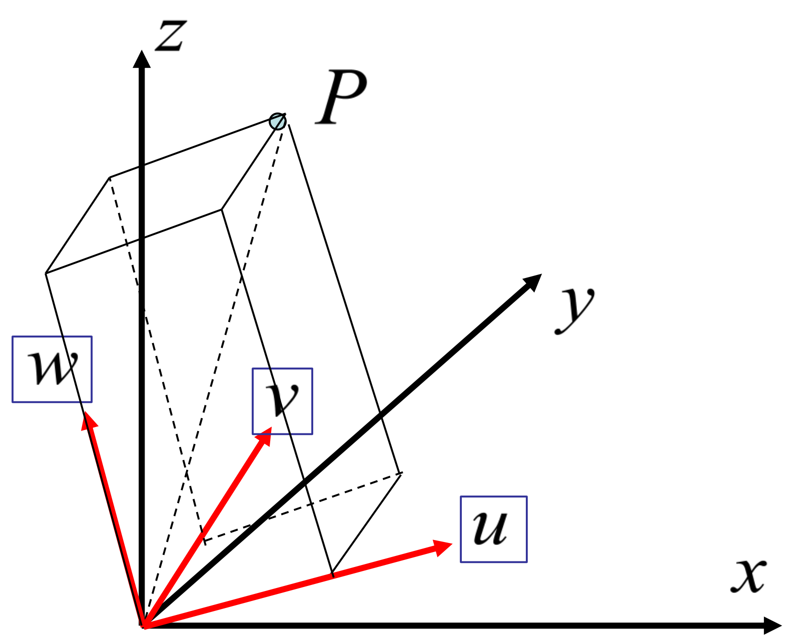
Preliminary

- Coordinate Transformation
  - Rotation only

$$P_{xyz} = p_x \mathbf{i}_x + p_y \mathbf{j}_y + p_z \mathbf{k}_z$$

$$P_{uvw} = p_u \mathbf{i}_u + p_v \mathbf{j}_v + p_w \mathbf{k}_w$$

$$P_{xyz} = R P_{uvw}$$



How to relate the coordinate in these two frames?

Preliminary

- Basic Rotation
  - p_x, p_y, and p_z represent the projections of P onto OX, OY, OZ axes, respectively
  - Since

$$P = p_u \mathrm{i_u} + p_v \mathrm{j_v} + p_w \mathrm{k_w}$$

$$p_x = \mathrm{i_x} \cdot P = \mathrm{i_x} \cdot \mathrm{i_u} p_u + \mathrm{i_x} \cdot \mathrm{j_v} p_v + \mathrm{i_x} \cdot \mathrm{k_w} p_w$$

$$p_y = \mathrm{j_y} \cdot P = \mathrm{j_y} \cdot \mathrm{i_u} p_u + \mathrm{j_y} \cdot \mathrm{j_v} p_v + \mathrm{j_y} \cdot \mathrm{k_w} p_w$$

$$p_z = \mathrm{k_z} \cdot P = \mathrm{k_z} \cdot \mathrm{i_u} p_u + \mathrm{k_z} \cdot \mathrm{j_v} p_v + \mathrm{k_z} \cdot \mathrm{k_w} p_w$$

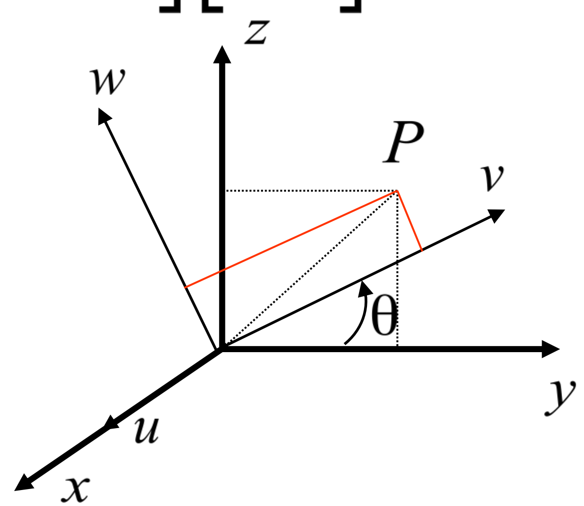
Preliminary

- Basic Rotation Matrix

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \mathrm{i_x} \cdot \mathrm{i_u} & \mathrm{i_x} \cdot \mathrm{j_v} & \mathrm{i_x} \cdot \mathrm{k_w} \\ \mathrm{j_y} \cdot \mathrm{i_u} & \mathrm{j_y} \cdot \mathrm{j_v} & \mathrm{j_y} \cdot \mathrm{k_w} \\ \mathrm{k_z} \cdot \mathrm{i_u} & \mathrm{k_z} \cdot \mathrm{j_v} & \mathrm{k_z} \cdot \mathrm{k_w} \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}$$

  - Rotation about x-axis with θ

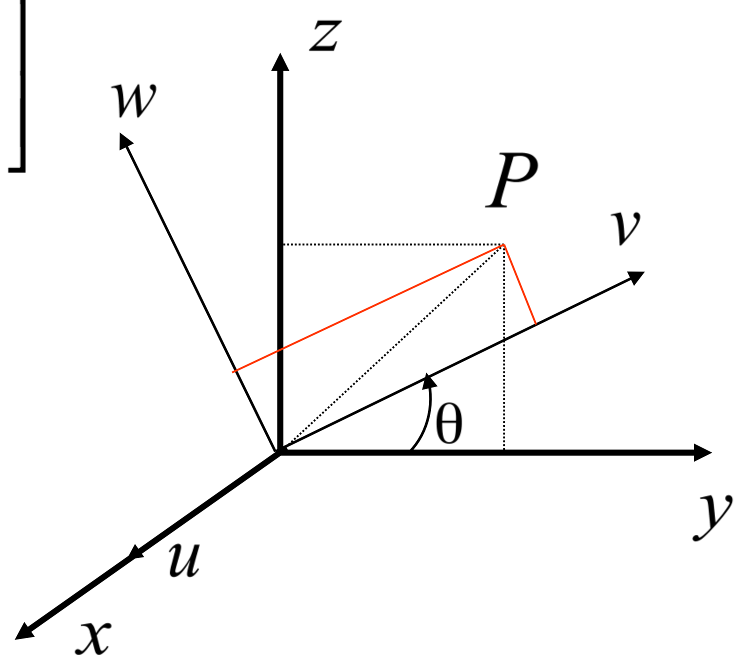
$$Rot(x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix}$$



Preliminary

- Is it True?
  - Rotation about x axis with θ

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}$$



$$p_x = p_u$$

$$p_y = p_v \cos\theta - p_w \sin\theta$$

$$p_z = p_v \sin\theta + p_w \cos\theta$$

Basic Rotation Matrices

- Rotation about x-axis with θ

$$Rot(x,\theta)=\begin{bmatrix}1 & 0 & 0\\ 0 & C\theta & -S\theta\\ 0 & S\theta & C\theta\end{bmatrix}$$

- Rotation about y-axis with θ

$$Rot(y,\theta)=\begin{bmatrix}C\theta & 0 & S\theta\\ 0 & 1 & 0\\ -S\theta & 0 & C\theta\end{bmatrix}$$

- Rotation about z-axis with θ

$$Rot(z,\theta)=\begin{bmatrix}C\theta & -S\theta & 0\\ S\theta & C\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$P_{xyz}=RP_{uvw}$$

Preliminary

- Basic Rotation Matrix

$$R = \begin{bmatrix} \mathrm{i_x \cdot i_u} & \mathrm{i_x \cdot j_v} & \mathrm{i_x \cdot k_w} \\ \mathrm{j_y \cdot i_u} & \mathrm{j_y \cdot j_v} & \mathrm{j_y \cdot k_w} \\ \mathrm{k_z \cdot i_u} & \mathrm{k_z \cdot j_v} & \mathrm{k_z \cdot k_w} \end{bmatrix}$$

$$P_{xyz} = RP_{uvw}$$

- Obtain the coordinate of P_{uvw} from the coordinate of P_{xyz}

Dot products are commutative!

$$\begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix} = \begin{bmatrix} \mathrm{i_u \cdot i_x} & \mathrm{i_u \cdot j_y} & \mathrm{i_u \cdot k_z} \\ \mathrm{j_v \cdot i_x} & \mathrm{j_v \cdot j_y} & \mathrm{j_v \cdot k_z} \\ \mathrm{k_w \cdot i_x} & \mathrm{k_w \cdot j_y} & \mathrm{k_w \cdot k_z} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

$$P_{uvw} = QP_{xyz}$$

$$Q = R^{-1} = R^T$$

$$QR = R^T R = R^{-1} R = I_3$$

<== 3X3 identity matrix

Example

- A point $a_{uvw} = (4,3,2)$ is attached to a rotating frame, the frame rotates 60 degree about the OZ axis of the reference frame. Find the coordinates of the point relative to the reference frame after the rotation.

$$a_{xyz} = Rot(z,60)a_{uvw}$$

$$= \begin{bmatrix} 0.5 & -0.866 & 0 \\ 0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.598 \\ 4.964 \\ 2 \end{bmatrix}$$

Example

- A point $a_{xyz} = (4,3,2)$ is the coordinate w.r.t. the reference coordinate system, find the corresponding point a_{uvw} w.r.t. the rotated OU-V-W coordinate system if it has been rotated 60 degree about OZ axis.

$$a_{uvw} = Rot(z,60)^T a_{xyz}$$

$$= \begin{bmatrix} 0.5 & 0.866 & 0 \\ -0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4.598 \\ -1.964 \\ 2 \end{bmatrix}$$

Composite Rotation Matrix

» A sequence of finite rotations

– matrix multiplications do not commute

– rules:

- if rotating coordinate O-U-V-W is rotating about principal axis of OXYZ frame, then ***Pre-multiply*** the previous (resultant) rotation matrix with an appropriate basic rotation matrix
- if rotating coordinate OUVW is rotating about its own principal axes, then ***post-multiply*** the previous (resultant) rotation matrix with an appropriate basic rotation matrix

Example

» Find the rotation matrix for the following operations:

Rotation ϕ about OY axis

Rotation θ about OW axis

Rotation α about OU axis

$$R = Rot(y,\phi) I_3 Rot(w,\theta) Rot(u,\alpha)$$

$$= \begin{bmatrix} C\phi & 0 & S\phi \\ 0 & 1 & 0 \\ -S\phi & 0 & C\phi \end{bmatrix} \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & -S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix}$$

Answer…

$$= \begin{bmatrix} C\phi C\theta & S\phi S\alpha - C\phi S\theta C\alpha & C\phi S\theta S\alpha + S\phi C\alpha \\ S\theta & C\theta C\alpha & -C\theta S\alpha \\ -S\phi C\theta & S\phi S\theta C\alpha + C\phi S\alpha & C\phi C\alpha - S\phi S\theta S\alpha \end{bmatrix}$$

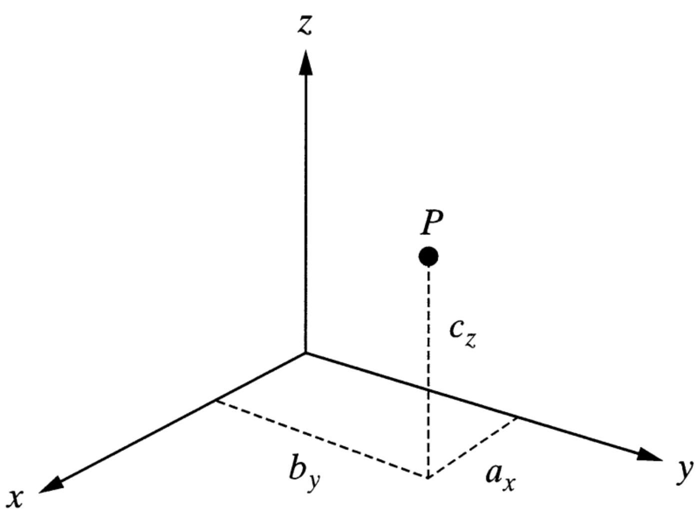
Pre-multiply if rotate about the OXYZ axes
Post-multiply if rotate about the OUVW axes

Matrix Representation

Representation of a Point in Space

A point P in space :

3 coordinates relative to a reference frame



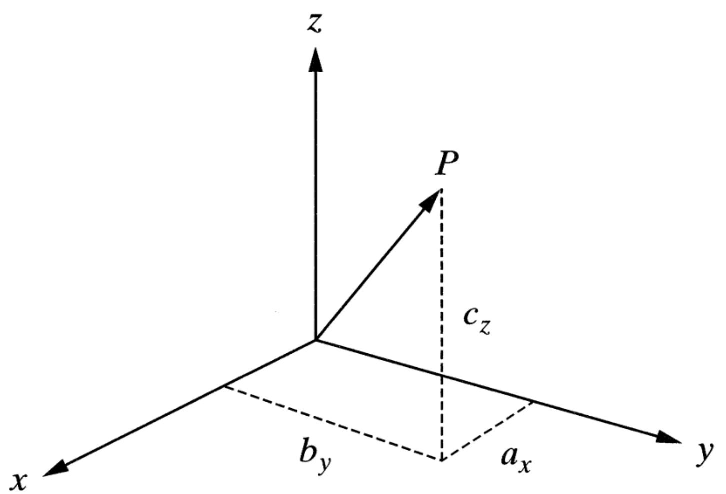
Representation of a point in space

$$P = a_x \hat{i} + b_y \hat{j} + c_z \hat{k}$$

Matrix Representation

Representation of a Vector in Space

A Vector P in space :
3 coordinates of its tail and of its head



Representation of a vector in space

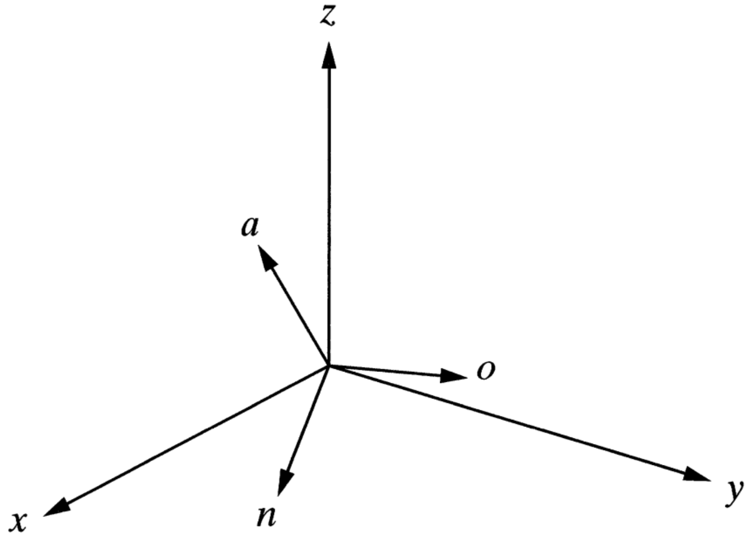
$$\overline{P} = a_x \hat{i} + b_y \hat{j} + c_z \hat{k}$$

$$\overline{P} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Matrix Representation

Representation of a Frame at the Origin of a Fixed-Reference Frame

Each Unit Vector is mutually perpendicular. :
normal, orientation, approach vector



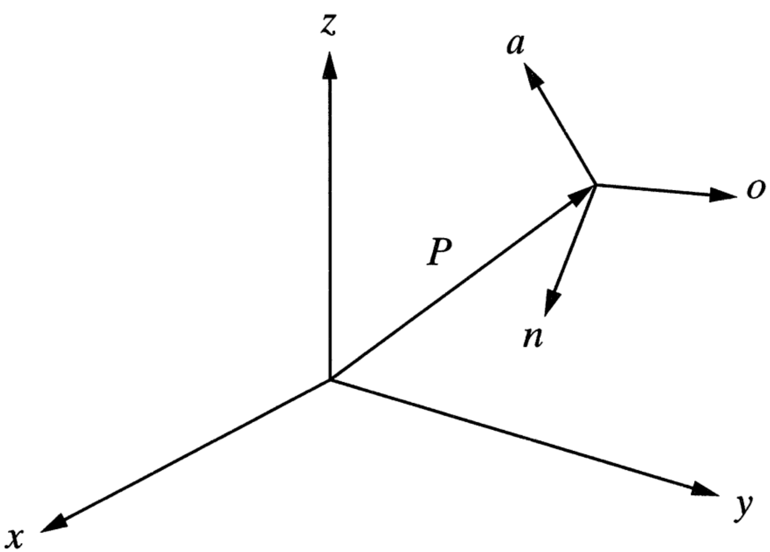
Representation of a frame at the origin of the reference frame

$$F = \begin{bmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{bmatrix}$$

Matrix Representation

Representation of a Frame in a Fixed Reference Frame

Each Unit Vector is mutually perpendicular. :
normal, orientation, approach vector



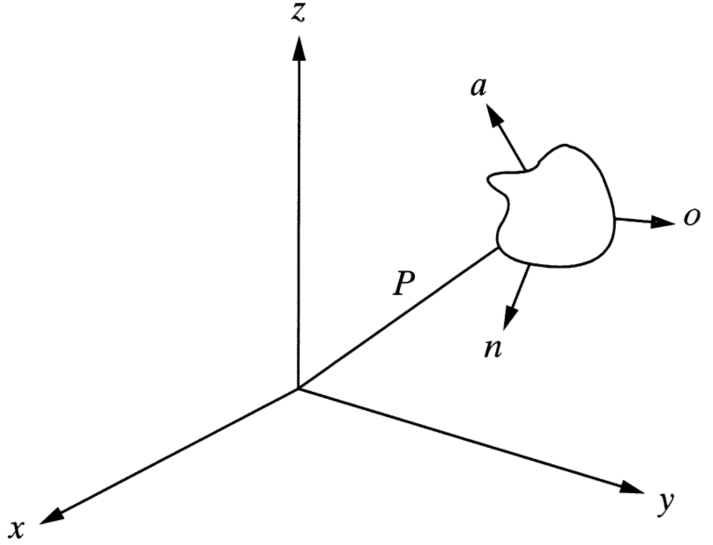
Representation of a frame in a frame

$$F = \begin{bmatrix} n_x & o_x & a_x & P_x \\ n_y & o_y & a_y & P_y \\ n_z & o_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix Representation

Representation of a Rigid Body

An object can be represented in space by attaching a frame to it and representing the frame in space.



Representation of an object in space

$$F_{object} = \begin{bmatrix} n_x & o_x & a_x & P_x \\ n_y & o_y & a_y & P_y \\ n_z & o_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Transformation

HOMOGENEOUS TRANSFORMATION MATRICES

A transformation matrices must be in square form.

- It is much easier to calculate the inverse of square matrices.
- To multiply two matrices, their dimensions must match.

$$F = \begin{bmatrix} n_x & o_x & a_x & P_x \\ n_y & o_y & a_y & P_y \\ n_z & o_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

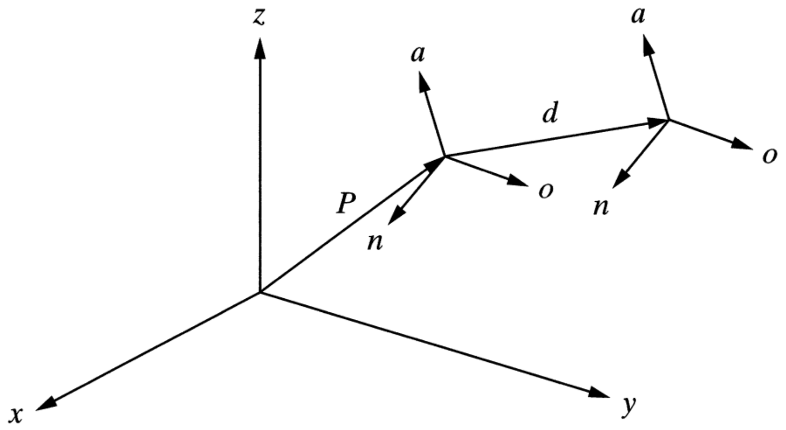
Homogeneous Transformation

REPRESENTATION OF TRANSFORMATIONS

Representation of a Pure Translation

A transformation is defined as making a movement in space.

- A pure translation.
- A pure rotation about an axis.
- A combination of translation or rotations.



$$T = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

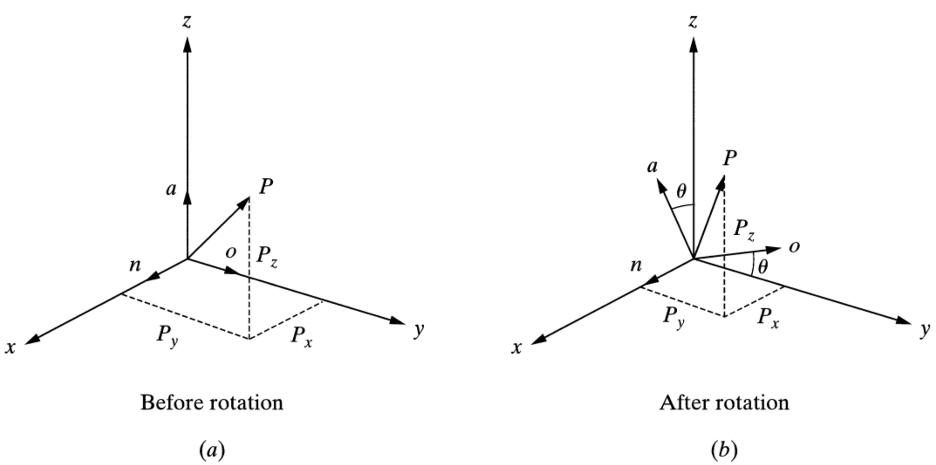
Representation of an pure translation in space

Homogeneous Transformation

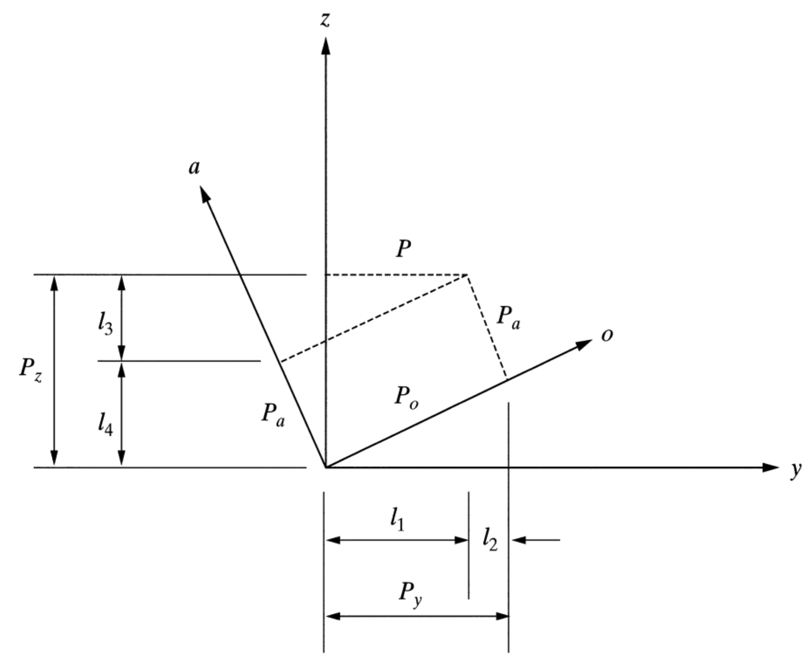
REPRESENTATION OF TRANSFORMATIONS

Representation of a Pure Rotation about an Axis

Assumption : The frame is at the origin of the reference frame and parallel to it.



Coordinates of a point in a rotating frame before and after rotation.



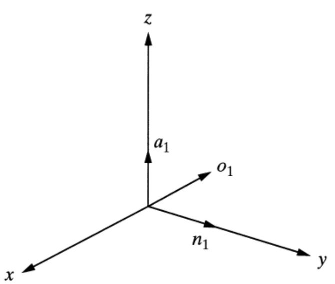
Coordinates of a point relative to the reference frame and rotating frame as viewed from the x-axis.

Homogeneous Transformation

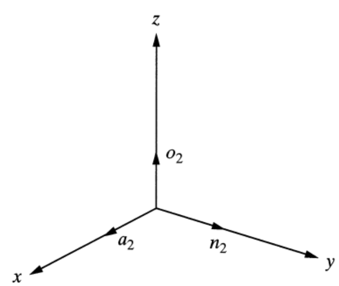
REPRESENTATION OF TRANSFORMATIONS

Representation of Combined Transformations

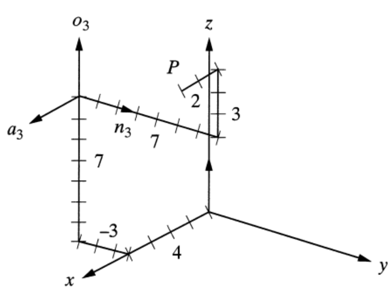
A number of successive translations and rotations….



After the first transformation

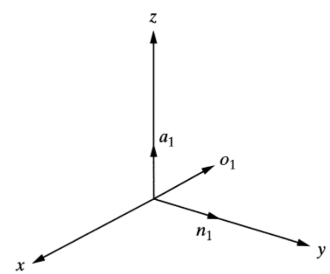


After the second transformation

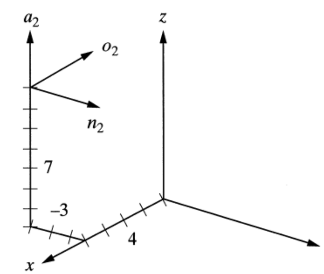


After the third transformation

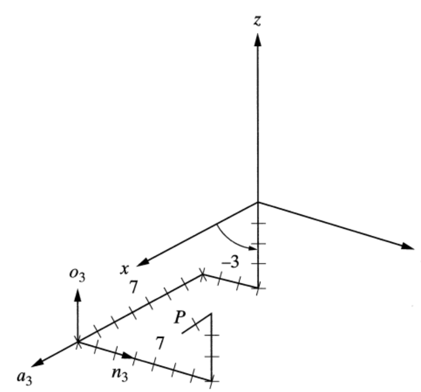
Effects of three successive transformations



After the first transformation



After the second transformation



After the third transformation

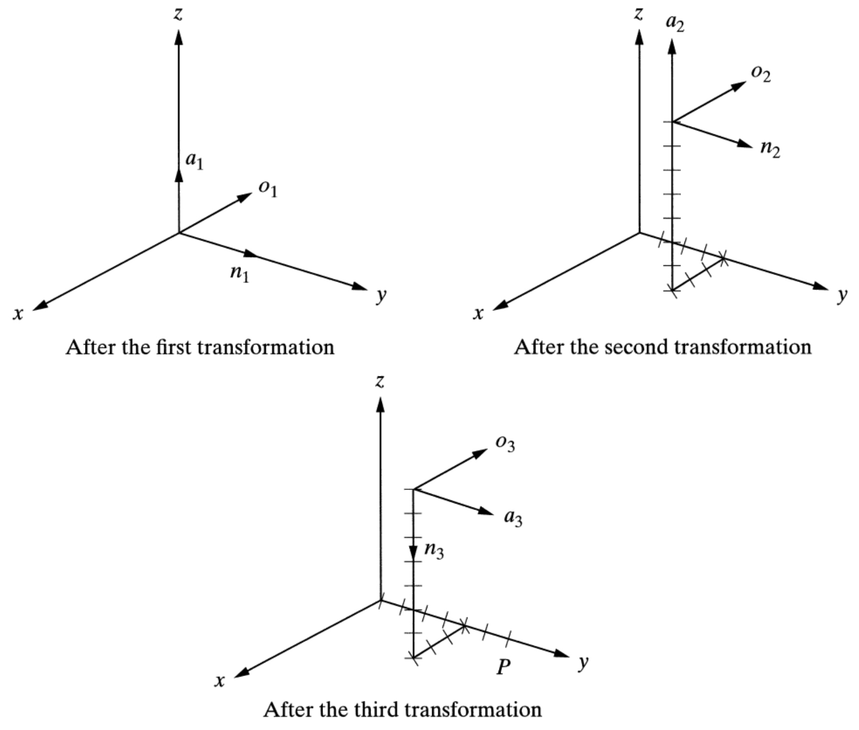
Changing the order of transformations will change the final result

Homogeneous Transformation

REPRESENTATION OF TRANSFORMATINS

Transformations Relative to the Rotating Frame

Example 2.8



Transformations relative to the current frames.

Thank you!