INDUSTRIAL AUTOMATION & ROBOTICS TECHNOLOGY

Orientation + Forward Kinematics

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Materials used

- Chapter 2,3 Introduction to Robotics, John J. Craig
- Chapter 2, Introduction to Robotics, Saeed B. Niku

Representation of Orientation

- All columns are mutually orthogonal and have unit magnitude.
- Determinant of a rotation matrix is equal to +1
- Rotation matrices may also be called proper orthonomal matrices, where proper refers to the fact that the determinant is +1
- Orientation can be describe with fewer than nine numbers

$$R = (I_3 - S)^{-1}(I_3 + S)$$

- I_3 $is\ a$ 3×3 identity matrix.
- S is a skew-symmetric matrix i.e., $S = -S^T$

$$S = \begin{bmatrix} 0 & -s_z & s_y \\ s_z & 0 & -s_x \\ -s_y & s_x & 0 \end{bmatrix}$$

Representation of Orientation

- Nine elements of a rotation matrix are not all indepndent

$$R = [\hat{X} \quad \hat{Y} \quad \hat{Z}]$$

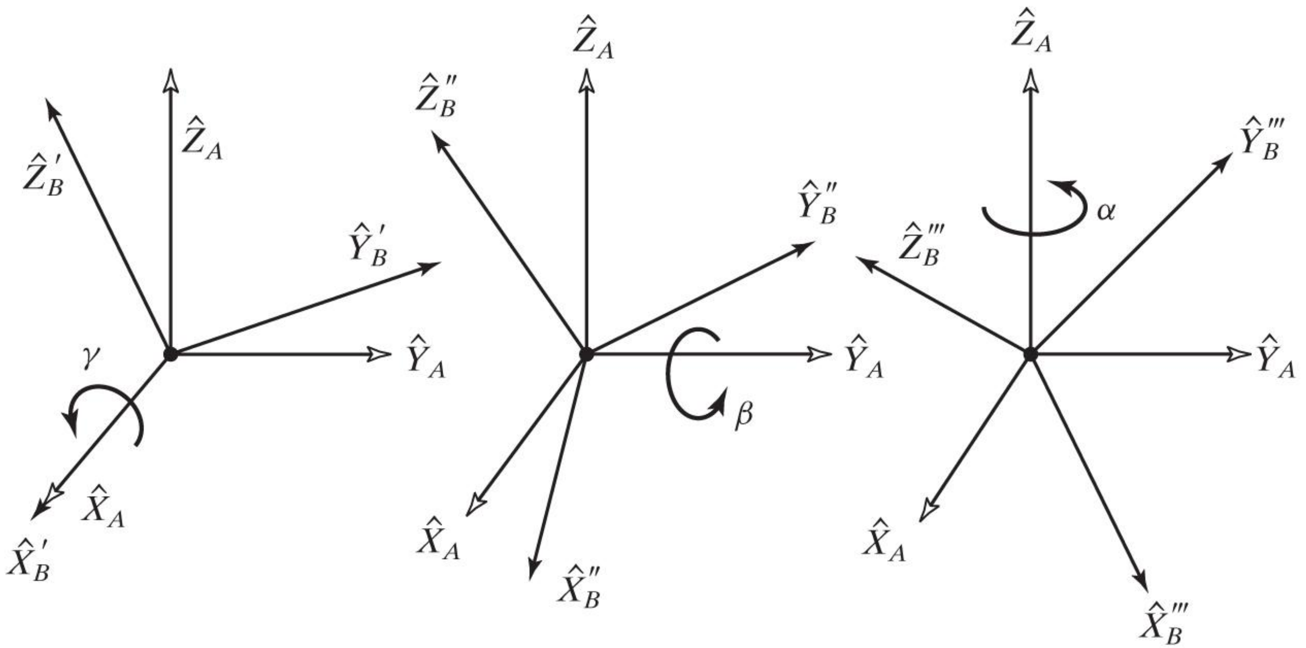
- There exists six constraints on the nine matrix elements

$$|\hat{X}| = 1, |\hat{Y}| = 1, |\hat{Z}| = 1$$

$$\hat{X} \cdot \hat{Y} = 0, \hat{X} \cdot \hat{Z} = 0, \hat{Y} \cdot \hat{Z} = 0$$

X-Y-Z Fixed Angles

Start with the frame coincident with a known reference frame {A}. Rotate {B} first about X, then about Y and finally about Z.



X-Y-Z Fixed Angles

- Sometimes this convention is referred to as roll, pitch, yaw angles

$$R = Rot(Z,\alpha) I_3 Rot(Y,\beta) Rot(X,\gamma)$$

$$= \begin{bmatrix} C\alpha & -S\alpha & 0 \\ S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{C}\beta & 0 & \mathrm{S}\beta \\ 0 & 1 & 0 \\ \text{-}\mathrm{S}\beta & 0 & \mathrm{C}\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\gamma & -S\gamma \\ 0 & S\gamma & C\gamma \end{bmatrix}$$

$$= \begin{bmatrix} C\alpha C\beta & C\alpha S\beta S\gamma - S\alpha C\gamma & C\alpha S\beta C\gamma + S\alpha S\gamma \\ S\alpha C\beta & S\alpha S\beta S\gamma + C\alpha C\gamma & S\alpha S\beta C\gamma - C\alpha S\gamma \\ -S\beta & C\beta S\gamma & C\beta C\gamma \end{bmatrix}$$

X-Y-Z Fixed Angles

- Inverse problem is of extraction of X-Y-Z fixed angles from a rotation matrix

$$R = \begin{bmatrix} C\alpha C\beta & C\alpha S\beta S\gamma - S\alpha C\gamma & C\alpha S\beta C\gamma + S\alpha S\gamma \\ S\alpha C\beta & S\alpha S\beta S\gamma + C\alpha C\gamma & S\alpha S\beta C\gamma - C\alpha S\gamma \\ -S\beta & C\beta S\gamma & C\beta C\gamma \end{bmatrix} \qquad R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- tan^{-1} is a two argument arc tangent function

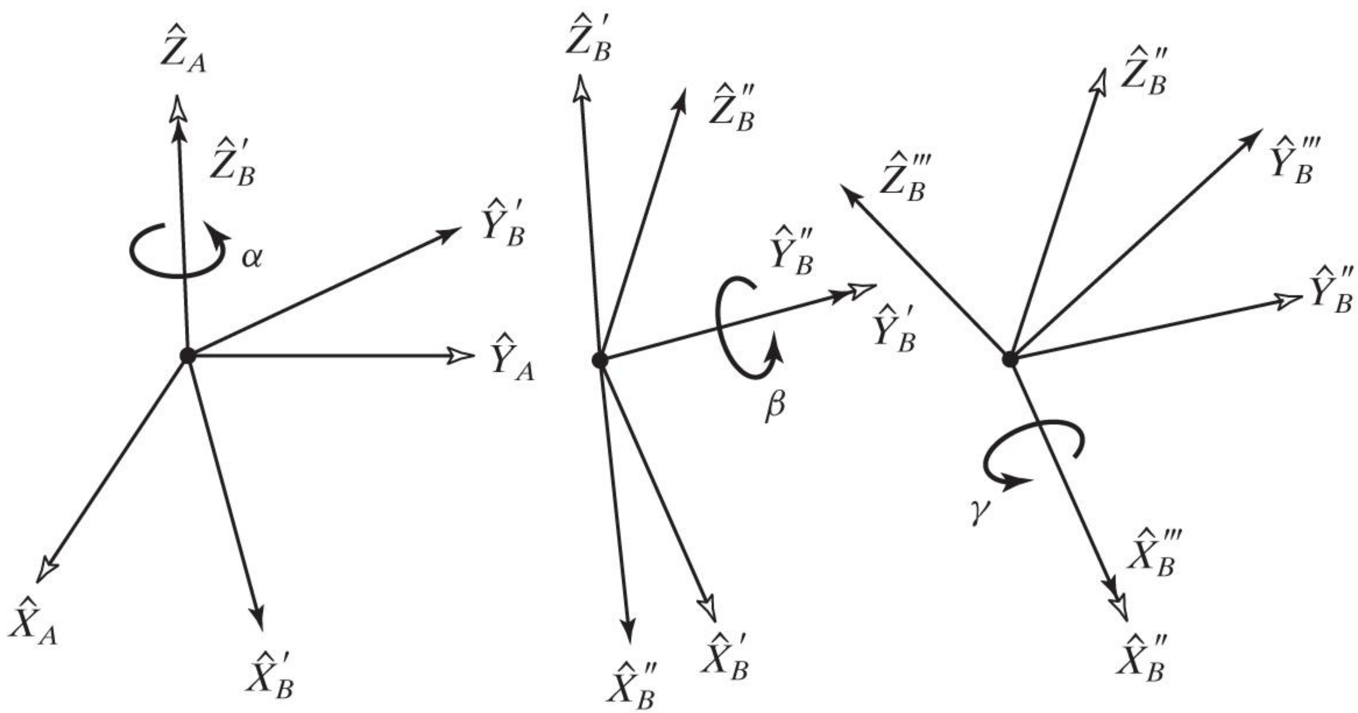
$$\alpha = arctan\left(\frac{r_{21}}{r_{11}}\right)$$

$$\beta = arctan\left(\frac{-r_{31}}{\sqrt{r_{11}^2 + r_{21}^2}}\right)$$

$$\gamma = arctan\left(\frac{r_{32}}{r_{33}}\right)$$

Z-Y-X Euler Angles

Start with the frame coincident with a known reference frame {A}.
Rotate {B} first about Z, then about Y and finally about X.



ZYX Euler Angles

Z-Y-X intrinsic rotation Euler angles are defined as follows:

- Rotate about Z (of the original fixed frame) by α (yaw)
- Rotate about Y of the new frame (frame after rotation in 1.) by β (pitch)
- Rotate about X of the new frame (frame after rotation in 2.) by γ (roll)

Z-Y-X Euler Angles

- Such sets of three rotations are called Euler Angles

$$R = Rot(\mathrm{Z},\alpha) I_3 Rot(\mathrm{Y},\beta) Rot(\mathrm{X},\gamma)$$

$$= \begin{bmatrix} C\alpha & -S\alpha & 0 \\ S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{C}\beta & 0 & \mathrm{S}\beta \\ 0 & 1 & 0 \\ \text{-}\mathrm{S}\beta & 0 & \mathrm{C}\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\gamma & -S\gamma \\ 0 & S\gamma & C\gamma \end{bmatrix}$$

$$= \begin{bmatrix} C\alpha C\beta & C\alpha S\beta S\gamma - S\alpha C\gamma & C\alpha S\beta C\gamma + S\alpha S\gamma \\ S\alpha C\beta & S\alpha S\beta S\gamma + C\alpha C\gamma & S\alpha S\beta C\gamma - C\alpha S\gamma \\ -S\beta & C\beta S\gamma & C\beta C\gamma \end{bmatrix}$$

Z pre-multiplied

Y and X post-multiplied

Z-Y-Z Euler Angles

Start with the frame coincident with a known reference frame {A}.
Rotate {B} first about Z, then about Y and finally about Z.

$$R = Rot(\mathrm{Z},\alpha) I_3 Rot(\mathrm{Y},\beta) Rot(\mathrm{Z},\gamma)$$

$$= \begin{bmatrix} C\alpha & -S\alpha & 0 \\ S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{C}\beta & 0 & \mathrm{S}\beta \\ 0 & 1 & 0 \\ \text{-}\mathrm{S}\beta & 0 & \mathrm{C}\beta \end{bmatrix} \begin{bmatrix} C\gamma & -S\gamma & 0 \\ S\gamma & C\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C\alpha C\beta C\gamma - S\alpha S\gamma & -C\alpha C\beta S\gamma - S\alpha C\gamma & C\alpha S\beta \\ S\alpha C\beta C\gamma + C\alpha S\gamma & -S\alpha C\beta S\gamma + C\alpha C\gamma & S\alpha S\beta \\ -S\beta C\gamma & S\beta S\gamma & C\beta \end{bmatrix}$$

Review

- Coordinate transformation from $\{B\}$ to $\{A\}$

$$ {}^{A}r^{P} = {}^{A}R_{B}\,{}^{B}r^{P} + {}^{A}r^{o'} $$

$$ \begin{bmatrix} {}^{A}r^{P} \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}r^{o'} \\ 0_{1\times3} & 1 \end{bmatrix} \begin{bmatrix} {}^{B}r^{P} \\ 1 \end{bmatrix} $$

- Homogeneous transformation matrix

$$ {}^{A}T_{B} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}r^{o'} \\ 0_{1\times3} & 1 \end{bmatrix} = \begin{bmatrix} R_{3\times3} & P_{3\times1} \\ 0 & 1 \end{bmatrix} $$

Rotation matrix

Position vector

Scaling

Review

- Homogeneous Transformation
  - Special cases

1. Translation

$$ {}^{A}T_{B} = \begin{bmatrix} I_{3\times3} & {}^{A}r^{o'} \\ 0_{1\times3} & 1 \end{bmatrix} $$

2. Rotation

$$ {}^{A}T_{B} = \begin{bmatrix} {}^{A}R_{B} & 0_{3\times1} \\ 0_{1\times3} & 1 \end{bmatrix} $$

Review

- Composite Homogeneous Transformation Matrix
- Rules:
  - Transformation (rotation/translation) w.r.t. (X,Y,Z) (OLD FRAME), using **pre-multiplication**
  - Transformation (rotation/translation) w.r.t. (U,V,W) (NEW FRAME), using **post-multiplication**

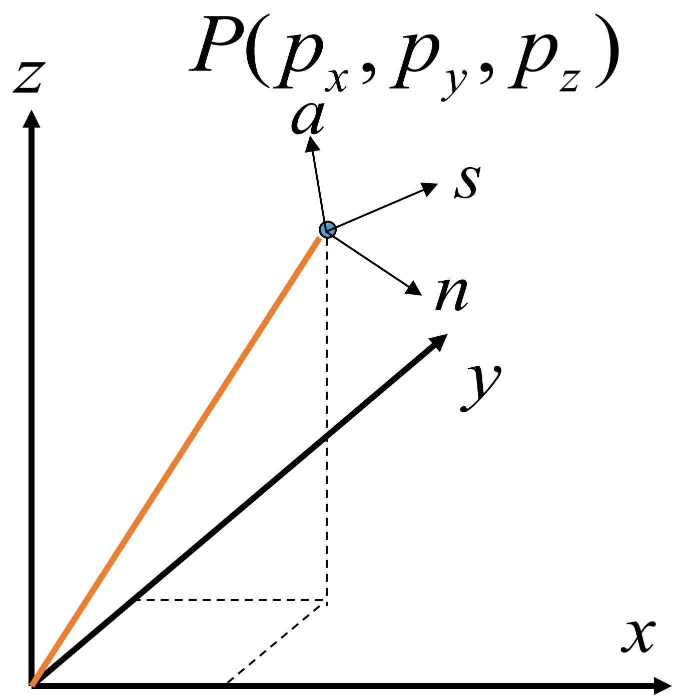
Review

- Homogeneous Representation
  - A point in R^3 space

$$P = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$ ← Homogeneous coordinate of P w.r.t. OXYZ

  - A frame in R^3 space

$$F = \begin{bmatrix} n & s & a & P \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

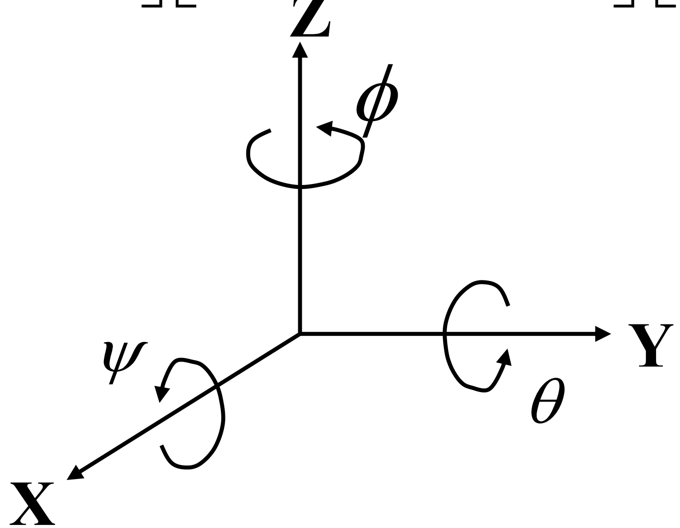


Quiz 1

- How to get the resultant rotation matrix for YPR?

$$T = R_{z,\phi} R_{y,\theta} R_{x,\psi}$$

$$= \begin{bmatrix} C\phi & -S\phi & 0 & 0 \\ S\phi & C\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta & 0 & S\theta & 0 \\ 0 & 1 & 0 & 0 \\ -S\theta & 0 & C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\psi & -S\psi & 0 \\ 0 & S\psi & C\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Quiz 2

- Geometric Interpretation?

- $$T = \begin{bmatrix} R_{3\times3} & P_{3\times1} \\ 0 & 1 \end{bmatrix}$$

Orientation of OUVW coordinate frame w.r.t. OXYZ frame

Position of the origin of OUVW coordinate frame w.r.t. OXYZ frame

Inverse Homogeneous Matrix?

$$T^{-1} = \begin{bmatrix} R^T & -R^T P \\ 0 & 1 \end{bmatrix}$$

Inverse of the rotation submatrix is equivalent to its transpose

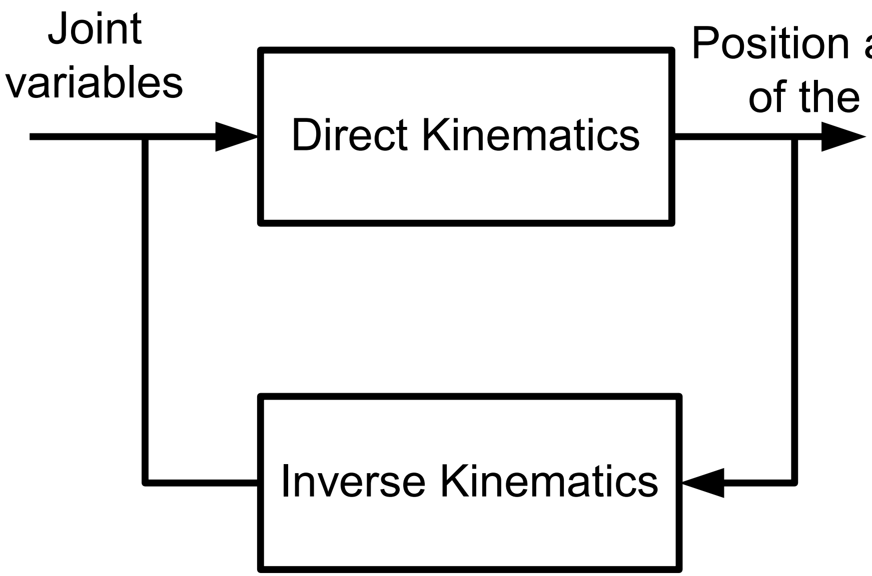
Position of the origin of OXYZ reference frame w.r.t. OUVW frame

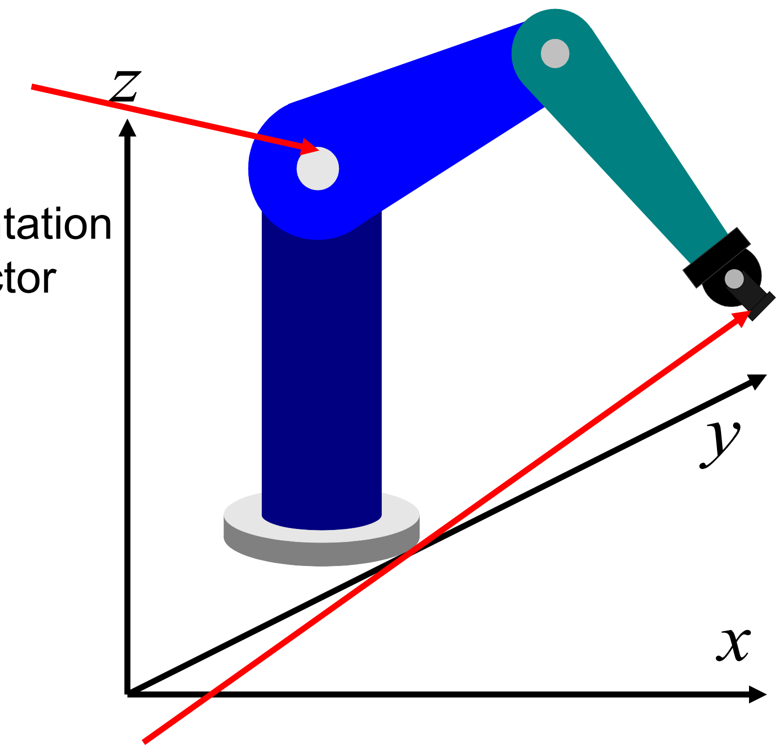
$$T^{-1}T = \begin{bmatrix} R^T & -R^T P \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R^T R & 0 \\ 0 & 1 \end{bmatrix} = I_{4\times4}$$

Kinematics Model

- Forward (direct) Kinematics

$$q = (q_1, q_2, \cdots q_n)$$

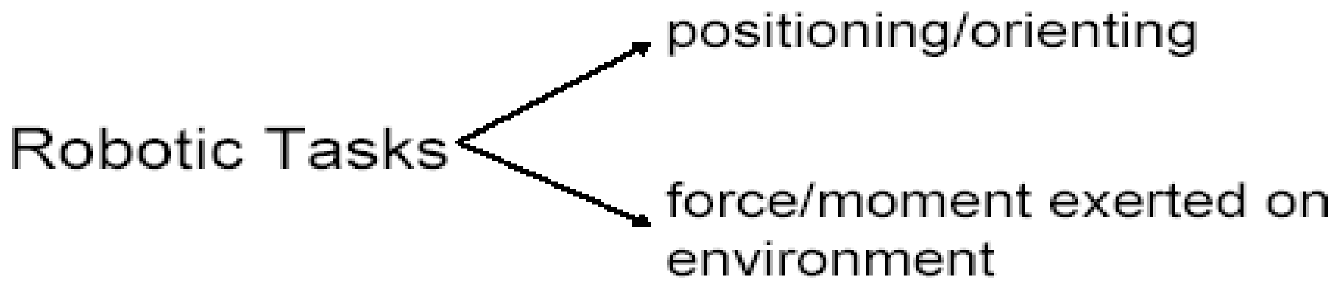




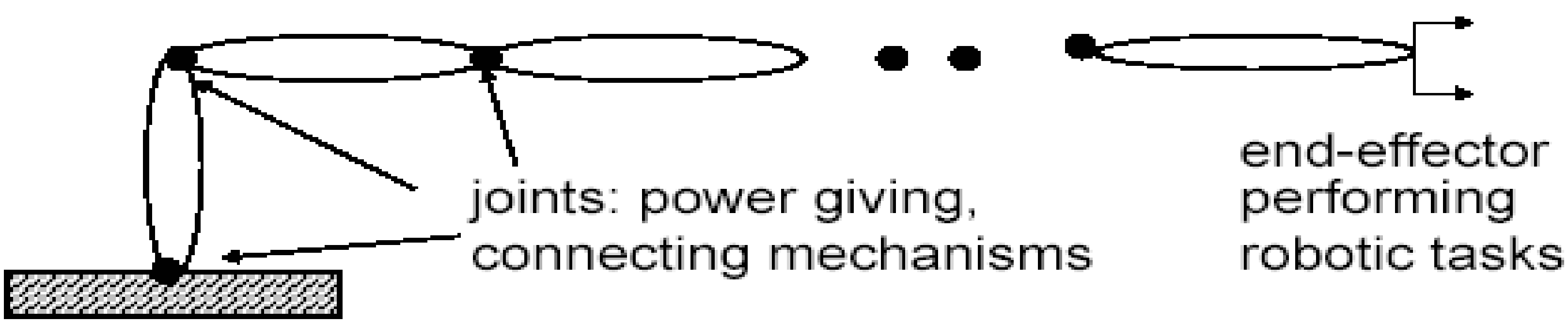
$$Y = (x, y, z, \phi, \theta, \psi)$$

- Inverse Kinematics

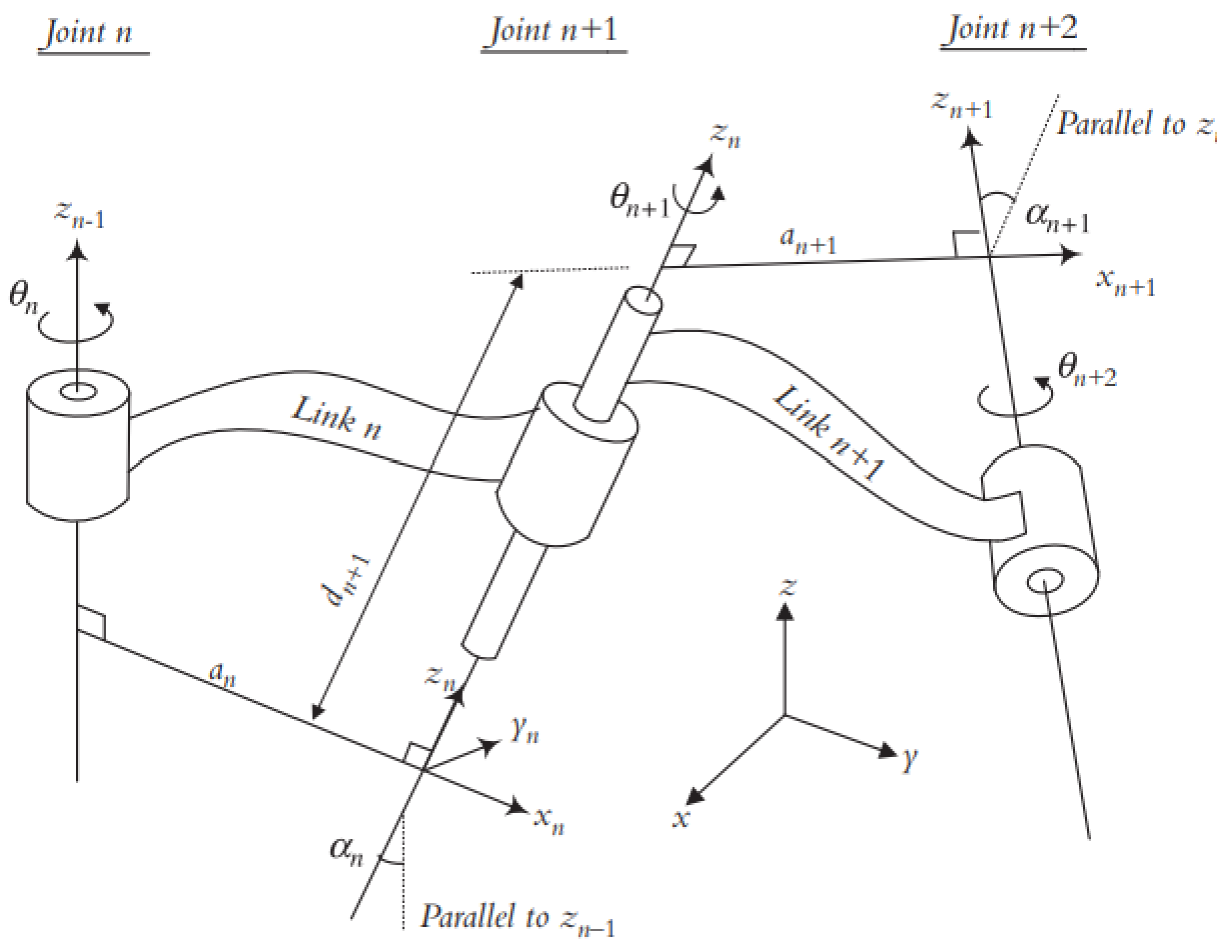
Robot Links and Joints



Chain of rigid bodies connected by joints



Denavit Hartenberg Representation of a general purpose joint-link combination



Link and Joint Parameters

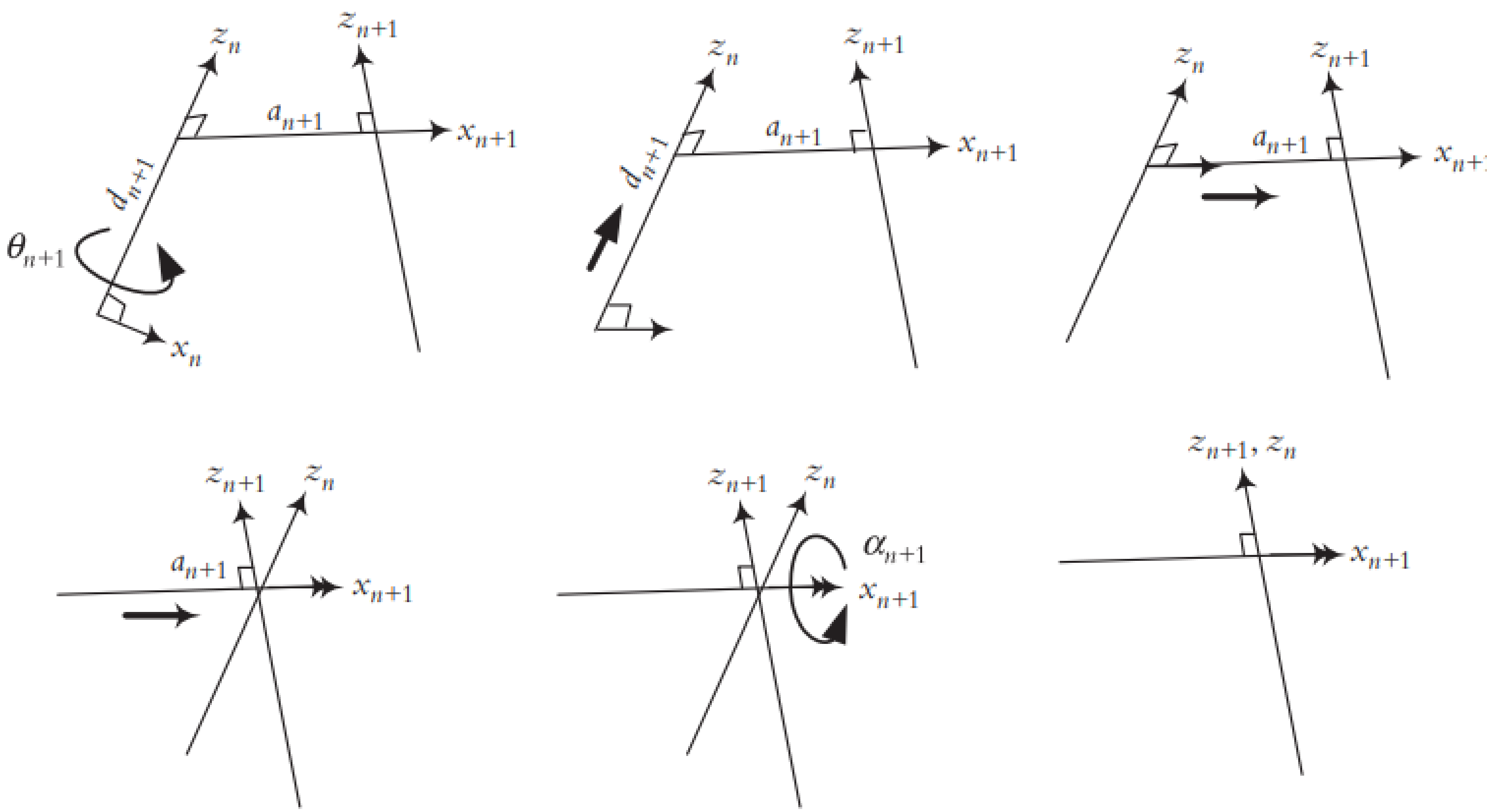
- *Joint angle* θ_i : the angle of rotation from the X_{i-1} axis to the X_i axis about the Z_{i-1} axis. It is the joint variable if joint i is rotary.

- *Joint distance* d_i: the distance from the origin of the (i-1) coordinate system to the intersection of the Z_{i-1} axis and the X_i axis along the Z_{i-1} axis. It is the joint variable if joint i is prismatic.

- *Link length* a_i: the distance from the intersection of the Z_{i-1} axis and the X_i axis to the origin of the ith coordinate system along the X_i axis.

- *Link twist angle* α_i: the angle of rotation from the Z_{i-1} axis to the Z_i axis about the X_i axis.

DH Convention

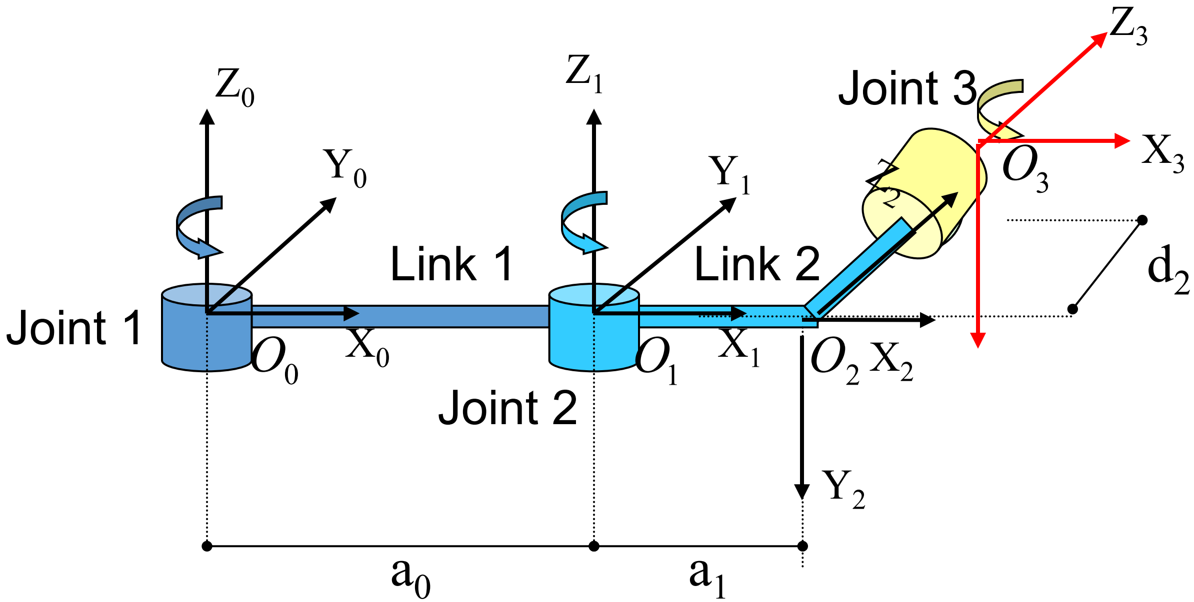


Denavit-Hartenberg Convention

- Number the joints from 1 to n starting with the base and ending with the end-effector.
- *Establish the base coordinate system.* Establish a right-handed orthonormal coordinate system (X_0, Y_0, Z_0) at the supporting base with Z_0 axis lying along the axis of motion of joint 1.
- *Establish joint axis.* Align the Z_i with the axis of motion (rotary or sliding) of joint i+1.
- *Establish the origin of the ith coordinate system.* Locate the origin of the ith coordinate at the intersection of the Z_i & Z_{i-1} or at the intersection of common normal between the Z_i & Z_{i-1} axes and the Z_i axis.
- *Establish X_i axis.* Establish $X_i = \pm(Z_{i-1} \times Z_i)/\|Z_{i-1} \times Z_i\|$ or along the common normal between the Z_{i-1} & Z_i axes when they are parallel.
- *Establish Y_i axis.* Assign $Y_i = +(Z_i \times X_i)/\|Z_i \times X_i\|$ to complete the right-handed coordinate system.
- Find the link and joint parameters

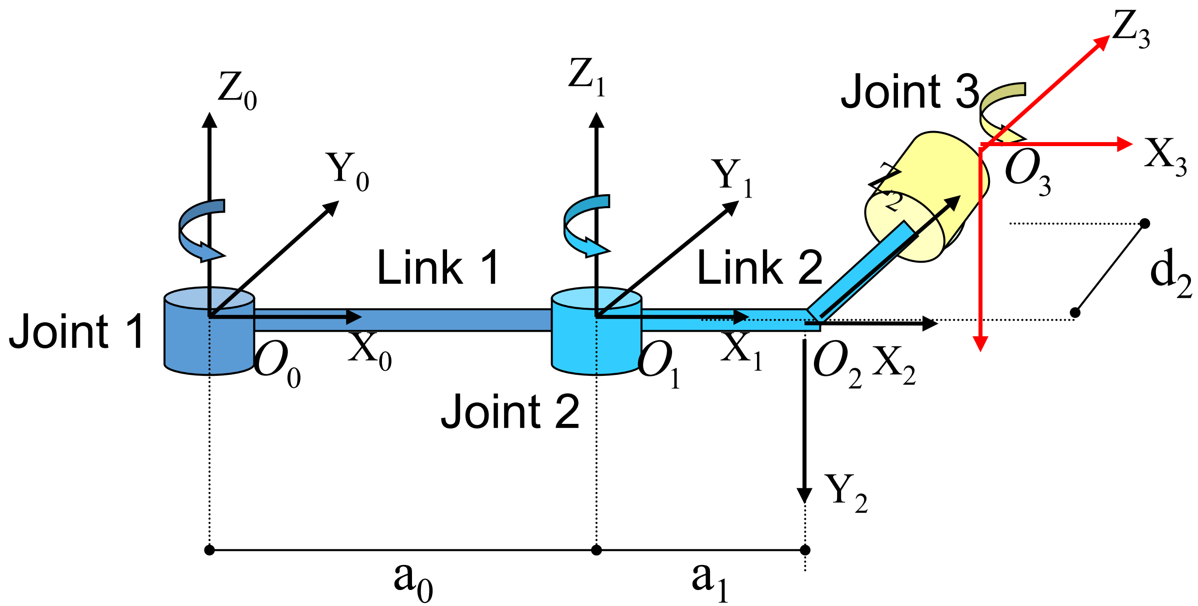
Example I

- 3 Revolute Joints



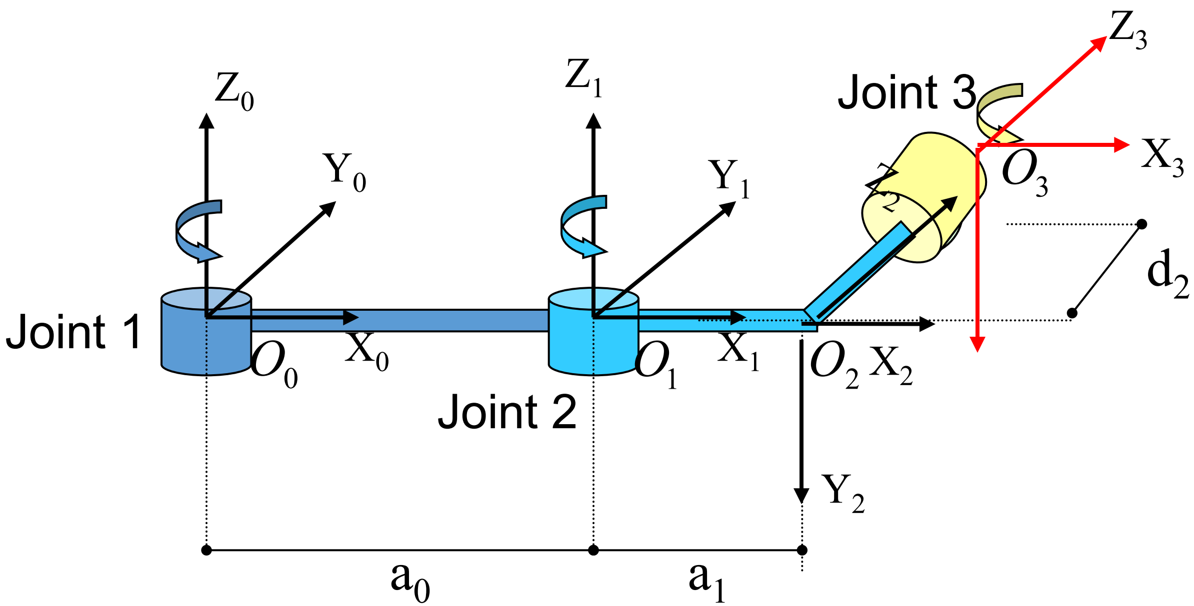
Link Coordinate Frames

- ***Assign Link Coordinate Frames:***
  - To describe the geometry of robot motion, we assign a Cartesian coordinate frame (O_i, X_i,Y_i,Z_i) to each link, as follows:
    - establish a right-handed orthonormal coordinate frame O_0 at the supporting base with Z_0 lying along joint 1 motion axis.
    - the Z_i axis is directed along the axis of motion of joint (i + 1), that is, link (i + 1) rotates about or translates along Z_i;



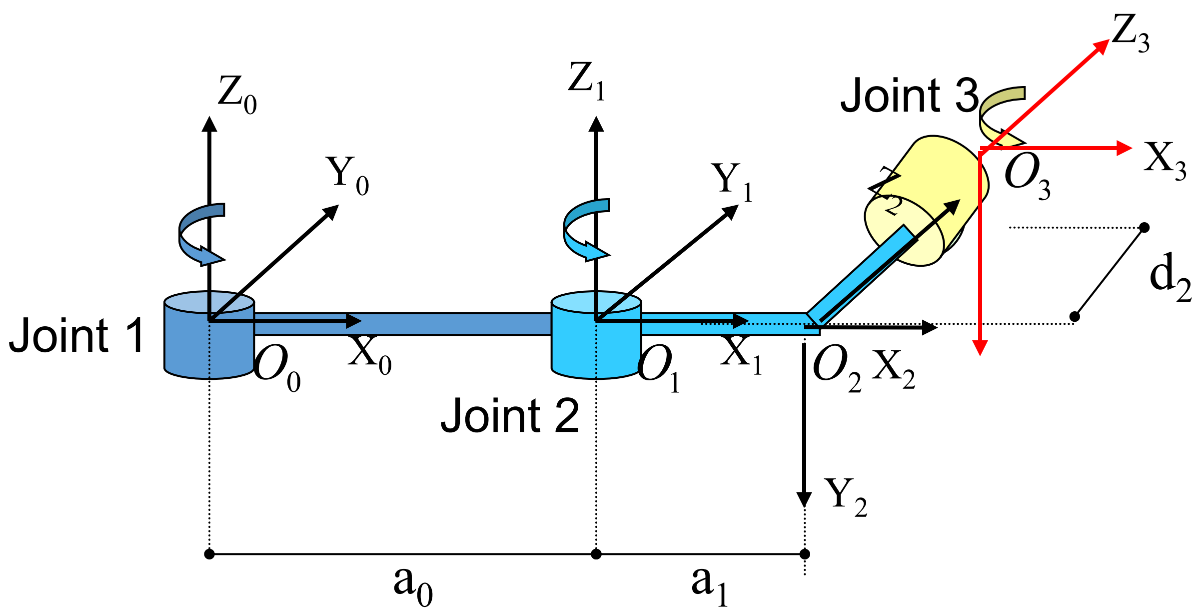
Link Coordinate Frames

- Locate the origin of the ith coordinate at the intersection of the Z_i & Z_{i-1} or at the intersection of common normal between the Z_i & Z_{i-1} axes and the Z_i axis.
- the X_i axis lies along the common normal from the Z_{i-1} axis to the Z_i axis $X_i = \pm(Z_{i-1} \times Z_i)/\|Z_{i-1} \times Z_i\|$, (if Z_{i-1} is parallel to Z_i, then X_i is specified arbitrarily, subject only to X_i being perpendicular to Z_i);

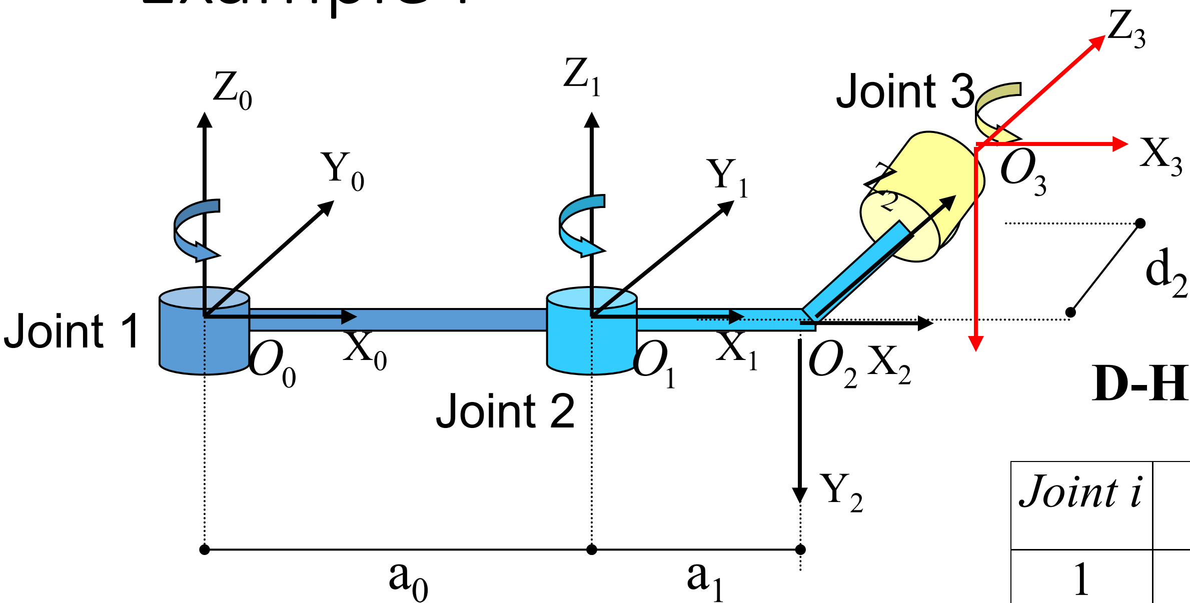


Link Coordinate Frames

- Assign $Y_i = +(Z_i \times X_i)/\|Z_i \times X_i\|$ to complete the right-handed coordinate system.
  - The hand coordinate frame is specified by the geometry of the end-effector. Normally, establish Z_n along the direction O^n of Z_{n-1} axis and pointing away from the robot; establish X_n such that it is normal to both Z_{n-1} and Z_n axes. Assign Y_n to complete the right-handed coordinate system.



Example I



D-H Link Parameter Table

Joint i	α_i	a_i	d_i	θ_i
1	0	a_0	0	θ_0
2	-90	a_1	0	θ_1
3	0	0	d_2	θ_2

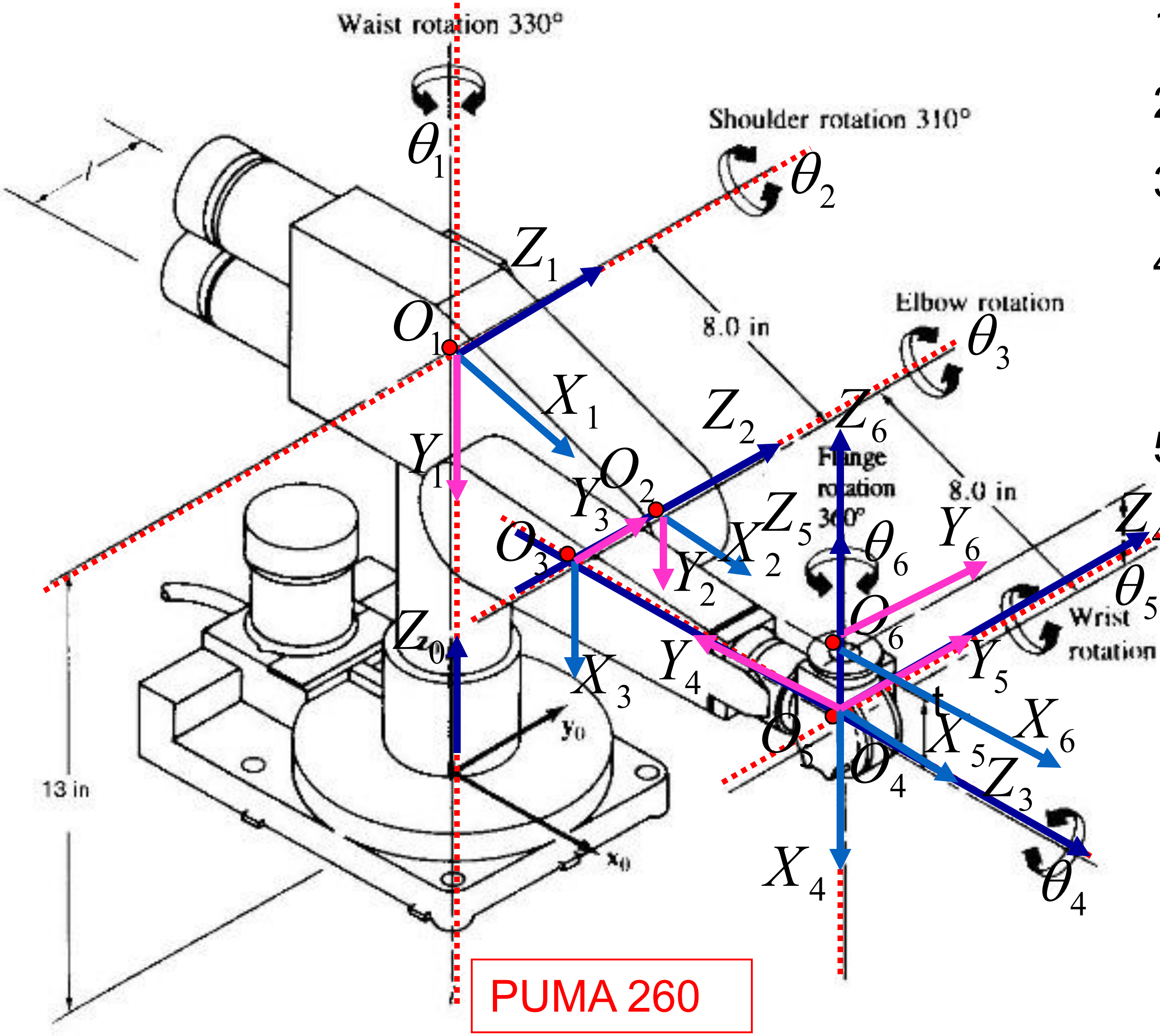
α_i : rotation angle from Z_{i-1} to Z_i about X_i

a_i : distance from intersection of Z_{i-1} & X_i to origin of i coordinate along X_i

d_i : distance from origin of (i-1) coordinate to intersection of Z_{i-1} & X_i along Z_{i-1}

θ_i : rotation angle from X_{i-1} to X_i about Z_{i-1}

Example II: PUMA 260

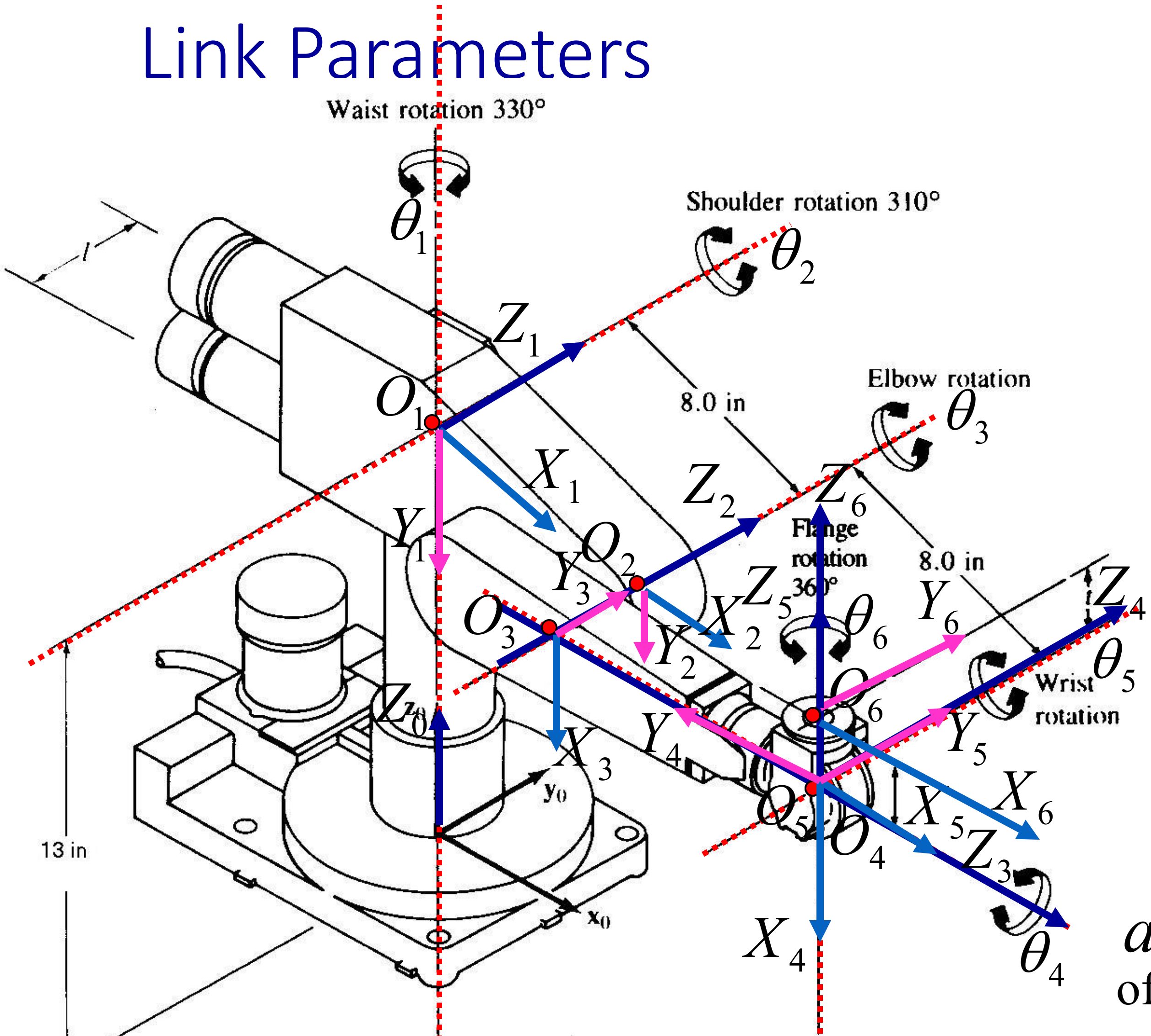


1. Number the joints
2. Establish base frame
3. Establish joint axis Z_i
4. Locate origin, (intersect. of Z_i & Z_{i-1}) OR (intersect of common normal & Z_i)
5. Establish X_i,Y_i

$$X_i = \pm(Z_{i-1} \times Z_i) / \|Z_{i-1} \times Z_i\|$$

$$Y_i = +(Z_i \times X_i) / \|Z_i \times X_i\|$$

Link Parameters



J	θ_i	α_i	a_i	d_i
1	θ_1	-90	0	13
2	θ_2	0	8	0
3	θ_3	90	0	-l
4	θ_4	-90	0	8
5	θ_5	90	0	0
6	θ_6	0	0	t

θ_i: angle from X_{i-1} to X_i about Z_{i-1}

α_i: angle from Z_{i-1} to Z_i about X_i

a_i: distance from intersection of Z_{i-1} & X_i to O_i along X_i

Joint distance d_i: distance from O_{i-1} to intersection of Z_{i-1} & X_i along Z_{i-1}

Transformation between i-1 and i

- Four successive elementary transformations are required to relate the *i*-th coordinate frame to the (*i*-1)-th coordinate frame:
  - Rotate about the Z_{i-1} axis an angle of θ_i to align the X_{i-1} axis with the X_i axis.
  - Translate along the Z_{i-1} axis a distance of d_i, to bring X_{i-1} and X_i axes into coincidence.
  - Translate along the X_i axis a distance of a_i to bring the two origins O_{i-1} and O_i as well as the X axis into coincidence.
  - Rotate about the X_i axis an angle of α_i (in the right-handed sense), to bring the two coordinates into coincidence.

Transformation between i-1 and i

- D-H transformation matrix for adjacent coordinate frames, *i* and *i-1.*
  - The position and orientation of the *i*-th frame coordinate can be expressed in the (*i-1*)th frame by the following homogeneous transformation matrix:

Source coordinate

Reference Coordinate

$$T_{i-1}^{i} = T(z_{i-1}, d_i) R(z_{i-1}, \theta_i) T(x_i, a_i) R(x_i, \alpha_i)$$

$$\begin{bmatrix} C\theta_i & -C\alpha_i S\theta_i & S\alpha_i S\theta_i & a_i C\theta_i \\ S\theta_i & C\alpha_i C\theta_i & -S\alpha_i C\theta_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Kinematic Equations

- Forward Kinematics
  - Given joint variables
  - End-effector position & orientation

$$q = (q_1, q_2, \cdots q_n)$$

$$\Downarrow$$

$$Y = (x, y, z, \phi, \theta, \psi)$$

- Homogeneous matrix T_0^n
  - specifies the location of the ith coordinate frame w.r.t. the base coordinate system
  - chain product of successive coordinate transformation matrices of T_{i-1}^i

$$T_0^n = T_0^1 T_1^2 \ldots T_{n-1}^n$$

$$= \begin{bmatrix} R_0^n & P_0^n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} n & s & a & P_0^n \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Orientation matrix (R_0^n)

Position vector (P_0^n)

Kinematics Equations

- Other representations
  - reference from, tool frame

$$T_{ref}^{tool} = B_{ref}^{0} T_{0}^{n} H_{n}^{tool}$$

  - Yaw-Pitch-Roll representation for orientation

$$T = R_{z,\phi} R_{y,\theta} R_{x,\psi}$$

$$= \begin{bmatrix} C\phi & -S\phi & 0 & 0 \\ S\phi & C\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta & 0 & S\theta & 0 \\ 0 & 1 & 0 & 0 \\ -S\theta & 0 & C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\psi & -S\psi & 0 \\ 0 & S\psi & C\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Representing forward kinematics

- Forward kinematics

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{bmatrix} \Rightarrow \begin{bmatrix} p_x \\ p_y \\ p_z \\ \phi \\ \theta \\ \varphi \end{bmatrix}$$

- Transformation Matrix

$$T = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Representing forward kinematics

- Yaw-Pitch-Roll representation for orientation

$$T_0^n = \begin{bmatrix} C\phi C\vartheta & C\phi S\theta S\psi - S\phi C\psi & C\phi S\theta C\psi + S\phi S\psi & p_x \\ S\phi C\theta & S\phi S\theta S\psi + C\phi C\psi & S\phi S\theta C\psi - C\phi S\psi & p_y \\ -S\theta & C\theta S\psi & C\theta C\psi & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_0^n = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

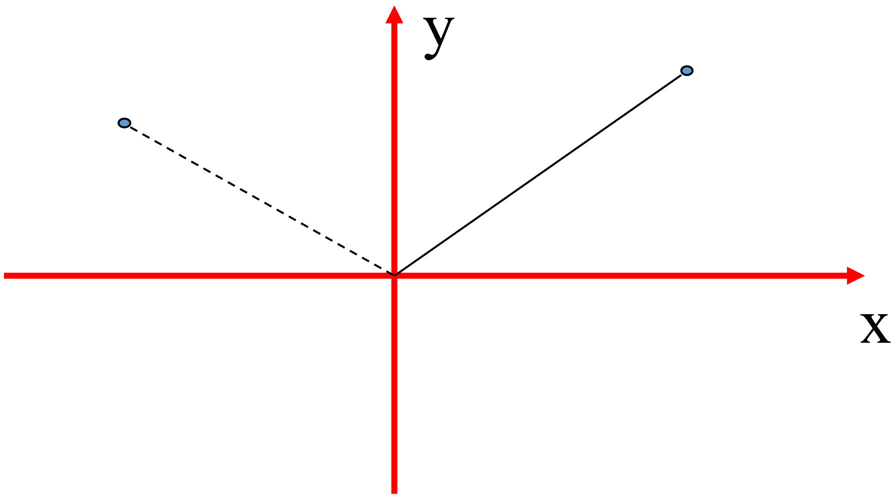
$$\theta = \sin^{-1}(-n_z)$$

$$\psi = \cos^{-1}\left(\frac{a_z}{\cos\theta}\right)$$

$$\phi = \cos^{-1}\left(\frac{n_x}{\cos\theta}\right)$$

Problem? Solution is inconsistent and ill-conditioned!!

atan2(y,x)



$$\theta = a\tan 2(y,x) = \begin{cases} 0^\circ \le \theta \le 90^\circ & \mathit{for} \quad +x \quad \mathit{and} \quad +y \\ 90^\circ \le \theta \le 180^\circ & \mathit{for} \quad -x \quad \mathit{and} \quad +y \\ -180^\circ \le \theta \le -90^\circ & \mathit{for} \quad -x \quad \mathit{and} \quad -y \\ -90^\circ \le \theta \le 0^\circ & \mathit{for} \quad +x \quad \mathit{and} \quad -y \end{cases}$$

Yaw-Pitch-Roll Representation

$$T = R_{z,\phi} R_{y,\theta} R_{x,\psi}$$

$$= \begin{bmatrix} C\phi & -S\phi & 0 & 0 \\ S\phi & C\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta & 0 & S\theta & 0 \\ 0 & 1 & 0 & 0 \\ -S\theta & 0 & C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\psi & -S\psi & 0 \\ 0 & S\psi & C\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} n_x & s_x & a_x & 0 \\ n_y & s_y & a_y & 0 \\ n_z & s_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Yaw-Pitch-Roll Representation

$$R_{z,\phi}^{-1}T = R_{y,\theta}R_{x,\psi}$$

$$\begin{bmatrix} C\phi & S\phi & 0 & 0 \\ -S\phi & C\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & s_x & a_x & 0 \\ n_y & s_y & a_y & 0 \\ n_z & s_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Equation A)

$$= \begin{bmatrix} C\theta & 0 & S\theta & 0 \\ 0 & 1 & 0 & 0 \\ -S\theta & 0 & C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\psi & -S\psi & 0 \\ 0 & S\psi & C\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Yaw-Pitch-Roll Representation

- Compare LHS and RHS of Equation A, we have:

$$-\sin\phi \cdot n_x + \cos\phi \cdot n_y = 0 \quad \Rightarrow \quad \phi = a\tan 2(n_y, n_x)$$

$$\begin{cases} \cos\phi \cdot n_x + \sin\phi \cdot n_y = \cos\theta \\ n_z = -\sin\theta \end{cases} \quad \Rightarrow \quad \theta = a\tan 2(-n_z, \cos\phi \cdot n_z + \sin\phi \cdot n_y)$$

$$\begin{cases} -\sin\phi \cdot s_x + \cos\phi \cdot s_y = \cos\psi \\ -\sin\phi \cdot a_x + \cos\phi \cdot a_y = -\sin\psi \end{cases}$$

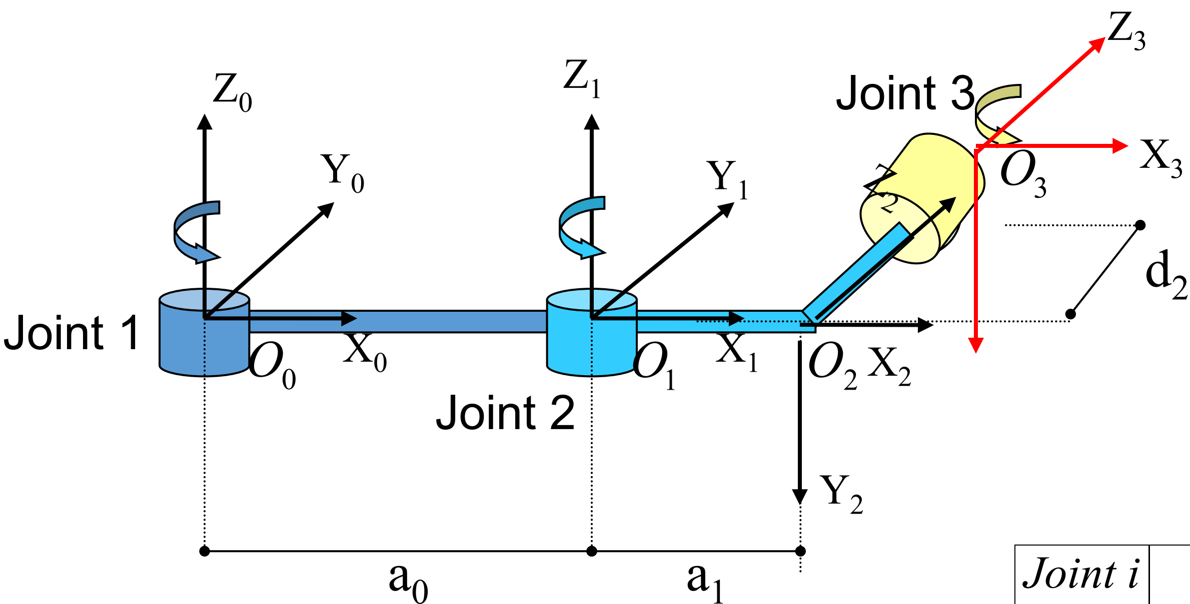
$$\Downarrow$$

$$\psi = a\tan 2(\sin\phi \cdot a_x - \cos\phi \cdot a_y, -\sin\phi \cdot s_x + \cos\phi \cdot s_y)$$

Kinematic Model

- Steps to derive kinematics model:
  - Assign D-H coordinates frames
  - Find link parameters
  - Transformation matrices of adjacent joints
  - Calculate Kinematics Matrix
  - When necessary, Euler angle representation

Example



Joint i	α_i	a_i	d_i	θ_i
1	0	a_0	0	θ_0
2	-90	a_1	0	θ_1
3	0	0	d_2	θ_2

Example

Joint i	α_i	a_i	d_i	θ_i
1	0	a_0	0	θ_0
2	-90	a_1	0	θ_1
3	0	0	d_2	θ_2

$$T_{i-1}^{i} = \begin{bmatrix} C\theta_i & -C\alpha_i S\theta_i & S\alpha_i S\theta_i & a_i C\theta_i \\ S\theta_i & C\alpha_i C\theta_i & -S\alpha_i C\theta_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

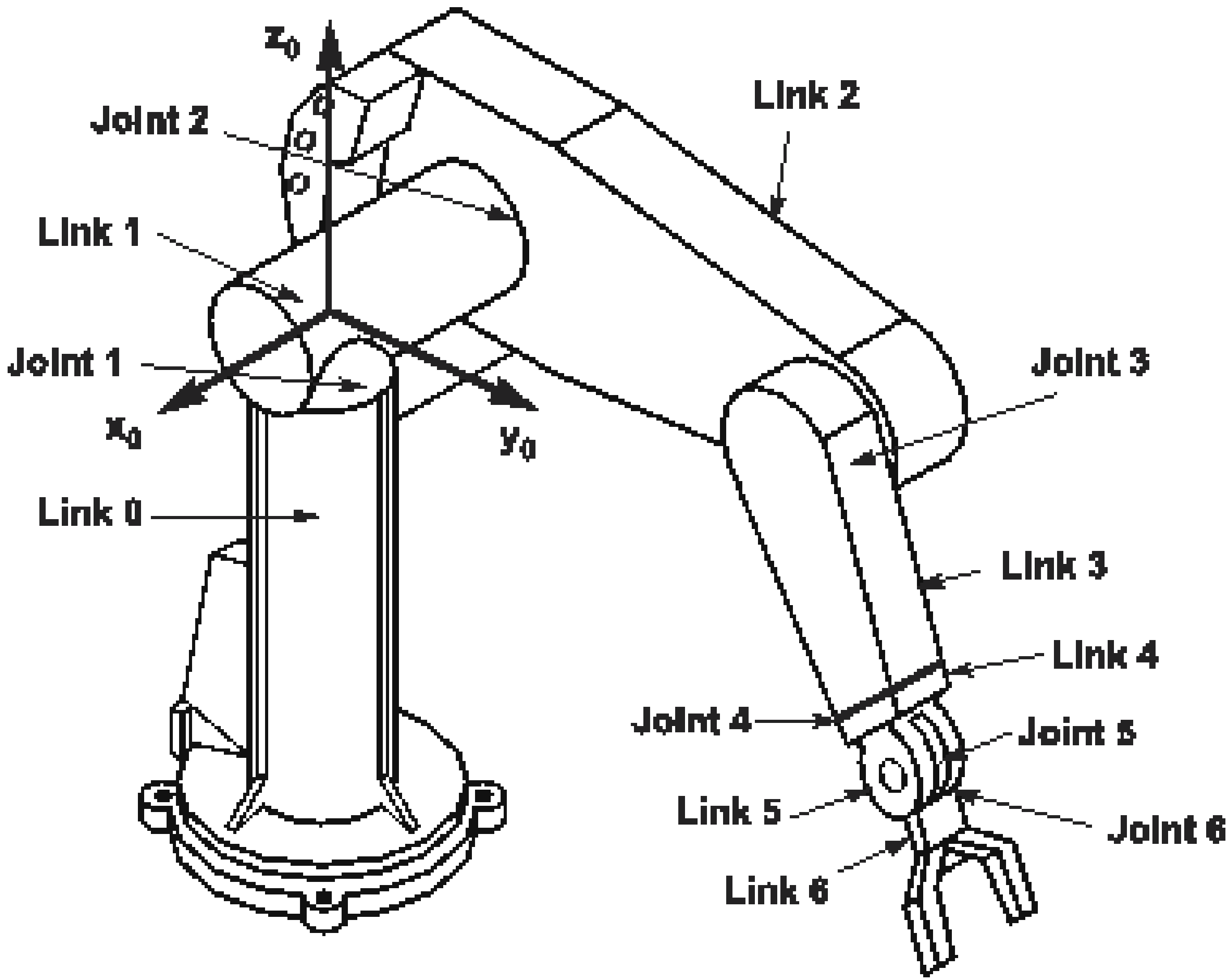
$$T_0^3 = (T_0^1)(T_1^2)(T_2^3)$$

$$T_0^1 = \begin{bmatrix} \cos\theta_0 & -\sin\theta_0 & 0 & a_0 \cos\theta_0 \\ \sin\theta_0 & \cos\theta_0 & 0 & a_0 \sin\theta_0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

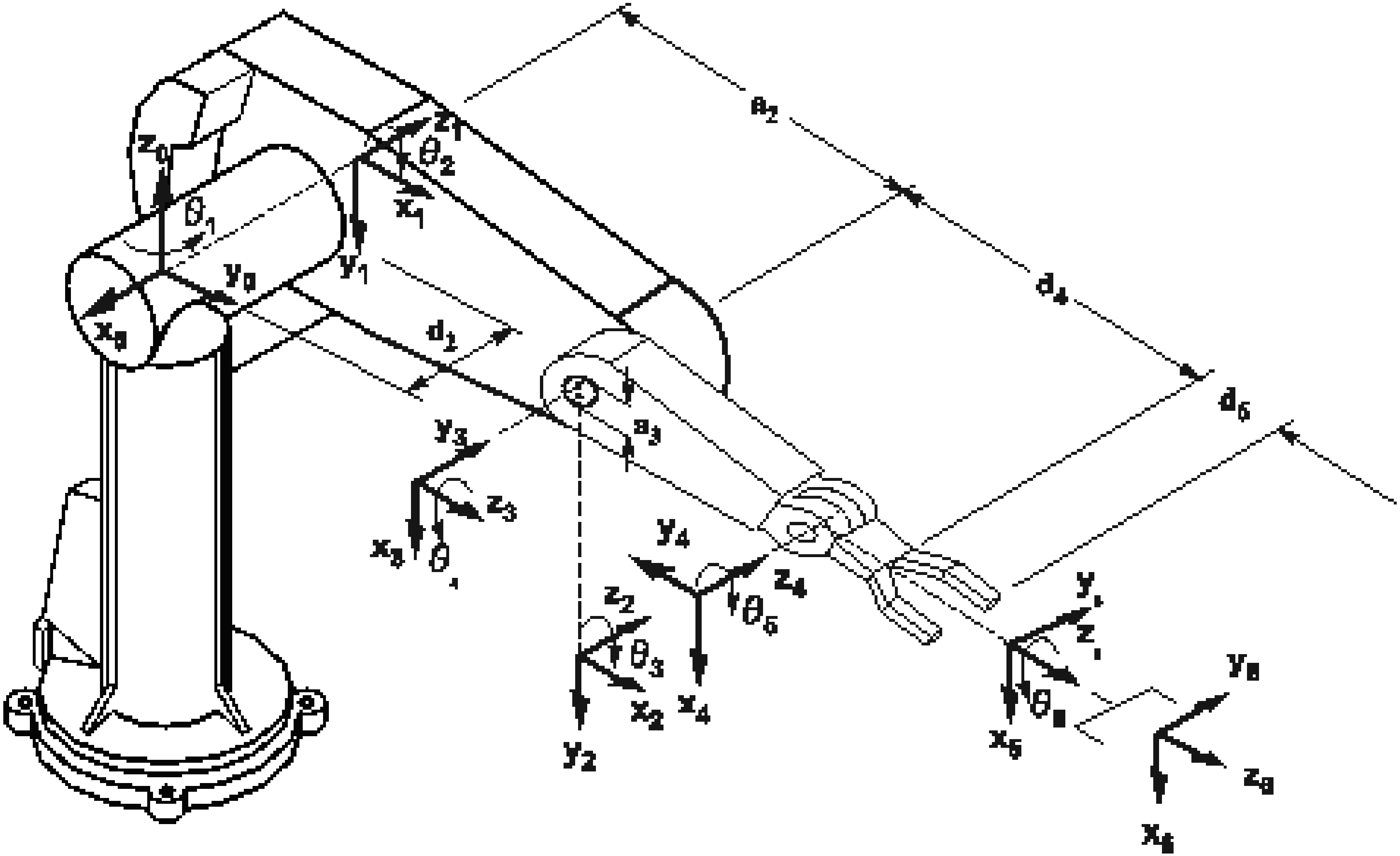
$$T_1^2 = \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & a_1 \cos\theta_1 \\ \sin\theta_1 & 0 & \cos\theta_1 & a_1 \sin\theta_1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^3 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: **Puma 560**



Example: Puma 560



Link Coordinate Parameters

PUMA 560 robot arm link coordinate parameters

Joint i	θ_i	α_i	$a_i(mm)$	$d_i(mm)$
1	θ_1	-90	0	0
2	θ_2	0	431.8	149.09
3	θ_3	90	-20.32	0
4	θ_4	-90	0	433.07
5	θ_5	90	0	0
6	θ_6	0	0	56.25

Example: **Puma 560**

$$
{}^0\boldsymbol{T}_1 = \begin{pmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^1\boldsymbol{T}_2 = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & a_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & a_2\sin\theta_2 \\ 0 & 0 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^2\boldsymbol{T}_3 = \begin{pmatrix} \cos\theta_3 & 0 & \sin\theta_3 & a_3\cos\theta_3 \\ \sin\theta_3 & 0 & -\cos\theta_3 & a_3\sin\theta_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^3\boldsymbol{T}_4 = \begin{pmatrix} \cos\theta_4 & 0 & -\sin\theta_4 & 0 \\ \sin\theta_4 & 0 & \cos\theta_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^4\boldsymbol{T}_5 = \begin{pmatrix} \cos\theta_5 & 0 & \sin\theta_5 & 0 \\ \sin\theta_5 & 0 & -\cos\theta_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^5\boldsymbol{T}_6 = \begin{pmatrix} \cos\theta_6 & -\sin\theta_6 & 0 & 0 \\ \sin\theta_6 & \cos\theta_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Example: **Puma 560**

$$
{}^{0}\boldsymbol{T}_6 = {}^{0}\boldsymbol{T}_1\, {}^{1}\boldsymbol{T}_2\, {}^{2}\boldsymbol{T}_3\, {}^{3}\boldsymbol{T}_4\, {}^{4}\boldsymbol{T}_5\, {}^{5}\boldsymbol{T}_6 = \begin{pmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

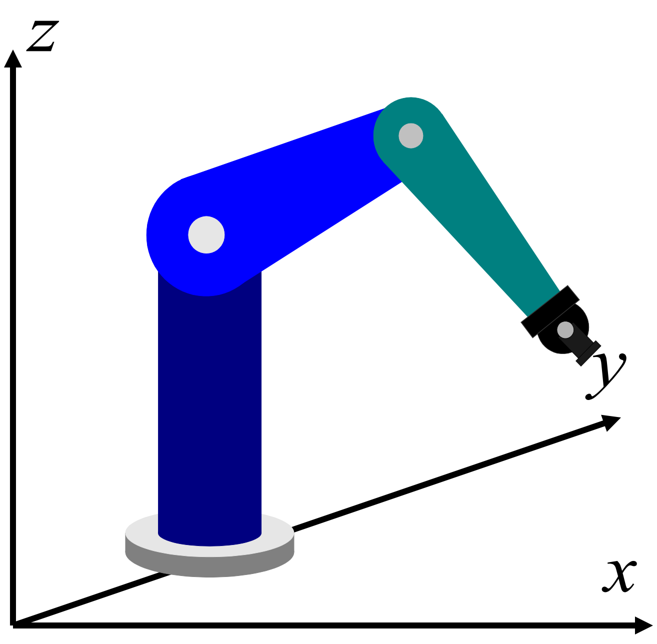
$$
\begin{aligned}
n_x &= c_1(c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6) - s_1(s_4c_5c_6 + c_4s_6) \\
n_y &= s_1(c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6) + c_1(s_4c_5c_6 + c_4s_6) \\
n_z &= -s_{23}(c_4c_5c_6 - s_4s_6) - c_{23}s_5c_6 \\
s_x &= c_1(-c_{23}(c_4c_5s_6 + s_4c_6) + s_{23}s_5s_6) - s_1(-s_4c_5s_6 + c_4c_6) \\
s_y &= s_1(-c_{23}(c_4c_5s_6 + s_4c_6) + s_{23}s_5s_6) + c_1(-s_4c_5s_6 + c_4c_6) \\
s_z &= s_{23}(c_4c_5s_6 + s_4c_6) - c_{23}s_5s_6 \\
a_x &= c_1(c_{23}c_4s_5 + s_{23}c_5) - s_1s_4s_5 \\
a_y &= s_1(c_{23}c_4s_5 + s_{23}c_5) + c_1s_4s_5 \\
a_z &= -s_{23}c_4s_5 + c_{23}c_5) \\
p_x &= c_1(d_6(c_{23}c_4s_5 + s_{23}c_5) + s_{23}d_4 + a_3c_{23} + a_2c_2) - s_1(d_6s_4s_5 + d_2) \\
p_y &= s_1(d_6(c_{23}c_4s_5 + s_{23}c_5) + s_{23}d_4 + a_3c_{23} + a_2c_2) + c_1(d_6s_4s_5 + d_2) \\
p_z &= d_6(c_{23}c_5 - s_{23}c_4s_5) + c_{23}d_4 - a_3s_{23} - a_2s_2
\end{aligned}
$$

Inverse Kinematics

- Given a desired position (P) & orientation (R) of the end-effector

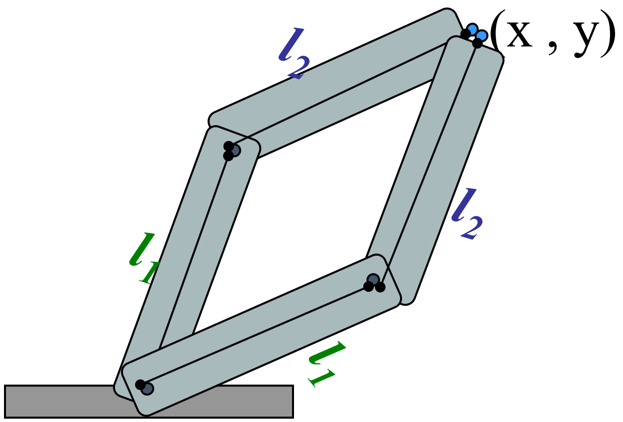
$$q = (q_1, q_2, \cdots q_n)$$

- Find the joint variables which can bring the robot the desired configuration



Inverse Kinematics

- More difficult
  - Systematic closed-form solution in general is not available
  - Solution not unique
    - Redundant robot
    - Elbow-up/elbow-down configuration
  - Robot dependent



Inverse Kinematics

- Transformation Matrix

$$T = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = T_0^1 T_1^2 T_2^3 T_3^4 T_4^5 T_5^6 \Rightarrow \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{bmatrix}$$

Special cases make the closed-form arm solution possible:

1. Three adjacent joint axes intersecting (PUMA, Stanford)
2. Three adjacent joint axes parallel to one another (MINIMOVER)

Thank you!