INDUSTRIAL AUTOMATION & ROBOTICS TECHNOLOGY

Differential Motions and Velocities

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Materials used

- Chapter 3, Introduction to Robotics, Saeed B. Niku

Differential Motions of a frame

- E 3.5: Find the effect of a differential rotation of 0.1 rad about the y-axis followed by a differential translation of [0.1, 0, 0.2] on the given frame B.

$$B = \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For $dx = 0.1, \quad dy = 0, \quad dz = 0.2, \quad \delta x = 0, \quad \delta y = 0.1, \quad \delta z = 0$

$$[dB] = [\Delta][B] = \begin{bmatrix} 0 & 0 & 0.1 & 0.1 \\ 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0.1 & 0 & 0.4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Interpretation of the Differential Change

- [dT] represents the changes in a frame as a result of differential motions.

$$dT = \begin{bmatrix} dn_x & do_x & da_x & dp_x \\ dn_y & do_y & da_y & dp_y \\ dn_z & do_z & da_z & dp_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{new} = dT + T_{old}$$

  - The frame is moved and rotated by the amount [dT]
  - {dn,do,da} components of [dT] = change in orientation
  - {dp} component of [dT] = change in position

Interpretation of the Differential Change

- E 3.6: Find the location and the orientation of frame B of Example 3.5 after the move.

$$B_{new} = B_{original} + dB$$

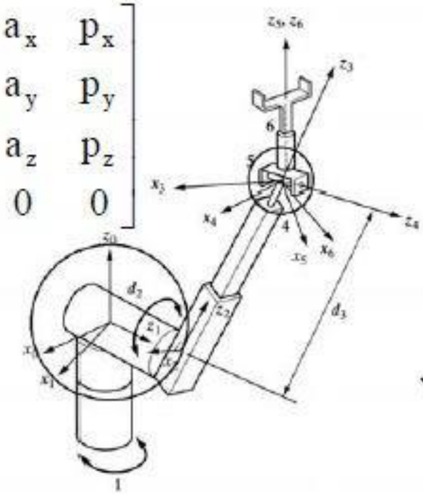
$$= \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0.1 & 0 & 0.4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.8 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.1 & 1 & 10.4 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & -0.1 & 2.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Differential Changes between Frames

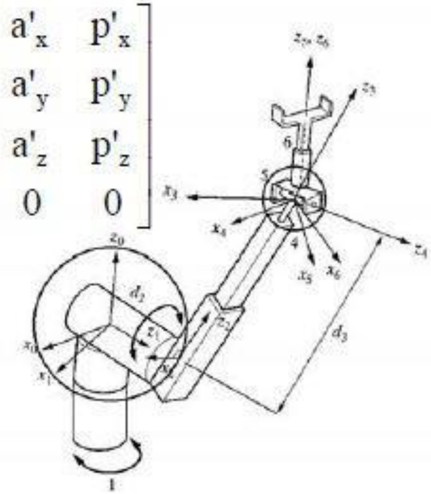
- Differential changes between frames
  - Diff. Op. Δ = diff. op. w.r.t. the fixed ref. frame = ${}^{U}\Delta$
  - Another Diff. Op. ${}^{T}\Delta$ = diff. op. w.r.t. the current frame
  - Therefore, postmultiply [T] by ${}^{T}\Delta$
  - $[dT] = [\Delta][T] = [T][{}^{T}\Delta] \rightarrow [T]^{-1}[\Delta][T] = [T]^{-1}[T][{}^{T}\Delta]$

$$[{}^{T}\Delta] = [T]^{-1}[\Delta][T]$$

$$T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



$$T + dT = \begin{bmatrix} n'_x & o'_x & a'_x & p'_x \\ n'_y & o'_y & a'_y & p'_y \\ n'_z & o'_z & a'_z & p'_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



Differential Changes between Frames

$$T^{-1} = \begin{bmatrix} n_x & n_y & n_z & -p \cdot n \\ o_x & o_y & o_z & -p \cdot o \\ a_x & a_y & a_z & -p \cdot a \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Delta = \begin{bmatrix} 0 & -\delta z & \delta y & dx \\ \delta z & 0 & -\delta x & dy \\ -\delta y & \delta x & 0 & dz \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$${}^{T}\Delta = T^{-1} \Delta T \Leftrightarrow$$

$$\begin{bmatrix} 0 & -{}^{T}\delta z & {}^{T}\delta y & {}^{T}dx \\ {}^{T}\delta z & 0 & -{}^{T}\delta x & {}^{T}dy \\ -{}^{T}\delta y & {}^{T}\delta x & 0 & {}^{T}dz \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} n_x & n_y & n_z & -p \cdot n \\ o_x & o_y & o_z & -p \cdot o \\ a_x & a_y & a_z & -p \cdot a \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\delta z & \delta y & dx \\ \delta z & 0 & -\delta x & dy \\ -\delta y & \delta x & 0 & dz \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} {}^{T}\delta x &= \boldsymbol{\delta} \cdot \mathbf{n} \\ {}^{T}\delta y &= \boldsymbol{\delta} \cdot \mathbf{o} \\ {}^{T}\delta z &= \boldsymbol{\delta} \cdot \mathbf{a} \\ {}^{T}dx &= \mathbf{n} \cdot [\boldsymbol{\delta} \times \mathbf{p} + \mathbf{d}] \\ {}^{T}dy &= \mathbf{o} \cdot [\boldsymbol{\delta} \times \mathbf{p} + \mathbf{d}] \\ {}^{T}dz &= \mathbf{a} \cdot [\boldsymbol{\delta} \times \mathbf{p} + \mathbf{d}] \end{aligned}$$

Differential Changes between Frames

- E 3.7: Find ${}^{B}\Delta$for Example 3.5.

$$\mathbf{n} = [0, 1, 0] \quad \mathbf{o} = [0, 0, 1] \quad \mathbf{a} = [1, 0, 0] \quad \mathbf{p} = [10, 5, 3]$$

$$\boldsymbol{\delta} = [0, 0.1, 0] \quad \mathbf{d} = [0.1, 0, 0.2]$$

$$\boldsymbol{\delta} \times \mathbf{p} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0.1 & 0 \\ 10 & 5 & 3 \end{vmatrix} = [0.3, 0, -1]$$

$$\boldsymbol{\delta} \times \mathbf{p} + \mathbf{d} = [0.3, 0, -1] + [0.1, 0, 0.2] = [0.4, 0, -0.8]$$

$${}^{B}dx = \mathbf{n} \cdot [\boldsymbol{\delta} \times \mathbf{p} + \mathbf{d}] = 0(0.4) + 1(0) + 0(-0.8) = 0$$

$${}^{B}dy = \mathbf{o} \cdot [\boldsymbol{\delta} \times \mathbf{p} + \mathbf{d}] = 0(0.4) + 0(0) + 1(-0.8) = -0.8$$

$${}^{B}dz = \mathbf{a} \cdot [\boldsymbol{\delta} \times \mathbf{p} + \mathbf{d}] = 1(0.4) + 0(0) + 0(-0.8) = 0.4$$

$${}^{B}\delta x = \boldsymbol{\delta} \cdot \mathbf{n} = 0(0) + 0.1(1) + 0(0) = 0.1$$

$${}^{B}\delta y = \boldsymbol{\delta} \cdot \mathbf{o} = 0(0) + 0.1(0) + 0(1) = 0$$

$${}^{B}\delta z = \boldsymbol{\delta} \cdot \mathbf{a} = 0(1) + 0.1(0) + 0(0) = 0$$

Differential Changes between Frames

$$[T^{-1}][\Delta][T] = {}^T\Delta = \begin{bmatrix} 0 & -{}^T\delta z & {}^T\delta y & {}^T dx \\ {}^T\delta z & 0 & -{}^T\delta x & {}^T dy \\ -{}^T\delta y & {}^T\delta x & 0 & {}^T dz \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$${}^B d = [0, -0.8, 0.4] \quad \text{and} \quad {}^B\delta = [0.1, 0, 0]$$

$${}^B\Delta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.8 \\ 0 & 0.1 & 0 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

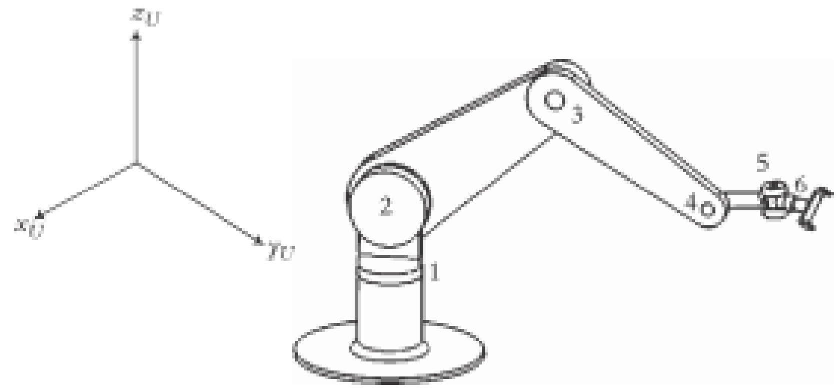
Differential Motions of a Robot and its Hand Frame

- Differential Motions
  - $dT = \Delta T = T^T \Delta$: changes in the components of n,o,a,p
  - Jacobian relates the joint micro-movements to the hand micro-movement.

$$\begin{bmatrix} dx \\ dy \\ dz \\ \delta x \\ \delta y \\ \delta z \end{bmatrix} = \begin{bmatrix} & & \\ & \text{Robot} & \\ & \text{Jacobian Matrix} & \\ & & \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \\ d\theta_4 \\ d\theta_5 \\ d\theta_6 \end{bmatrix} \quad \text{or} \quad [D] = [J][D_\theta]$$

Calculation of the Jacobian

- 6 dof articulate robot arm



- Arm matrix

$$
{}^{R}T_{\mathrm{H}} = A_1A_2A_3A_4A_5A_6
$$

$$
= \begin{bmatrix}
C_1(C_{234}C_5C_6 - S_{234}S_6) - S_1S_5C_6 & C_1(-C_{234}C_5S_6 - S_{234}C_6) + S_1S_5S_6 & C_1(C_{234}S_5) + S_1C_5 & C_1(C_{234}a_4 + C_{23}a_3 + C_2a_2) \\
S_1(C_{234}C_5C_6 - S_{234}S_6) + C_1S_5C_6 & S_1(-C_{234}C_5S_6 - S_{234}C_6) - C_1S_5S_6 & S_1(C_{234}S_5) - C_1C_5 & S_1(C_{234}a_4 + C_{23}a_3 + C_2a_2) \\
S_{234}C_5C_6 + C_{234}S_6 & -S_{234}C_5S_6 + C_{234}C_6 & S_{234}S_5 & S_{234}a_4 + S_{23}a_3 + S_2a_2 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

Calculation of the Jacobian

- $dp_x = dx$: use the chain rule

$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} c_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \\ s_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \\ s_{234}a_4 + s_{23}a_3 + s_2a_2 \\ 1 \end{bmatrix}$$

$$p_x = c_1(c_{234}a_4 + c_{23}a_3 + c_2a_2)$$

$$\begin{aligned} dp_x &= \frac{\partial p_x}{\partial \theta_1} d\theta_1 + \frac{\partial p_x}{\partial \theta_2} d\theta_2 + \cdots + \frac{\partial p_x}{\partial \theta_6} d\theta_6 \\ &= -s_1(c_{234}a_4 + c_{23}a_3 + c_2a_2)d\theta_1 + c_1[-s_{234}a_4 - s_{23}a_3 - s_2a_2]d\theta_2 \\ &\quad + c_1[-s_{234}a_4 - s_{23}a_3]d\theta_3 + c_1[-s_{234}a_4]d\theta_4 \end{aligned}$$

$\therefore$ the 1st row of Jacobian :

$$\frac{\partial p_x}{\partial \theta_1} = J_{11} = -s_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \qquad \frac{\partial p_x}{\partial \theta_4} = J_{14} = c_1[-s_{234}a_4]$$

$$\frac{\partial p_x}{\partial \theta_2} = J_{12} = c_1[-s_{234}a_4 - s_{23}a_3 - s_2a_2] \qquad \frac{\partial p_x}{\partial \theta_5} = J_{15} = 0$$

$$\frac{\partial p_x}{\partial \theta_3} = J_{13} = c_1[-s_{234}a_4 - s_{23}a_3] \qquad \frac{\partial p_x}{\partial \theta_6} = J_{16} = 0$$

Calculation of the Jacobian

- $dp_y = dy$: use the chain rule

$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} c_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \\ s_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \\ s_{234}a_4 + s_{23}a_3 + s_2a_2 \\ 1 \end{bmatrix}$$

$$p_y = s_1(c_{234}a_4 + c_{23}a_3 + c_2a_2)$$

$$\begin{aligned} dp_y &= \frac{\partial p_y}{\partial \theta_1} d\theta_1 + \frac{\partial p_y}{\partial \theta_2} d\theta_2 + \cdots + \frac{\partial p_y}{\partial \theta_6} d\theta_6 \\ &= c_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) d\theta_1 + s_1[-s_{234}a_4 - s_{23}a_3 - s_2a_2] d\theta_2 \\ &\quad + s_1[-s_{234}a_4 - s_{23}a_3] d\theta_3 + s_1[-s_{234}a_4] d\theta_4 \end{aligned}$$

$\therefore$ the 2^{nd} row of Jacobian :

$$\frac{\partial p_y}{\partial \theta_1} = J_{21} = c_1(c_{234}a_4 + c_{23}a_3 + c_2a_2)$$

$$\frac{\partial p_y}{\partial \theta_2} = J_{22} = s_1[-s_{234}a_4 - s_{23}a_3 - s_2a_2]$$

$$\frac{\partial p_y}{\partial \theta_3} = J_{23} = s_1[-s_{234}a_4 - s_{23}a_3]$$

$$\frac{\partial p_y}{\partial \theta_4} = J_{24} = s_1[-s_{234}a_4]$$

$$\frac{\partial p_y}{\partial \theta_5} = J_{25} = 0$$

$$\frac{\partial p_y}{\partial \theta_6} = J_{26} = 0$$

Calculation of the Jacobian

- $dp_z = dz$: use the chain rule

$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} c_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \\ s_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \\ s_{234}a_4 + s_{23}a_3 + s_2a_2 \\ 1 \end{bmatrix}$$

$$p_z = s_{234}a_4 + s_{23}a_3 + s_2a_2$$

$$\begin{aligned} dp_z &= \frac{\partial p_z}{\partial \theta_1} d\theta_1 + \frac{\partial p_z}{\partial \theta_2} d\theta_2 + \cdots + \frac{\partial p_z}{\partial \theta_6} d\theta_6 \\ &= [c_{234}a_4 + c_{23}a_3 + c_2a_2] d\theta_2 + [c_{234}a_4 + c_{23}a_3] d\theta_3 + [c_{234}a_4] d\theta_4 \end{aligned}$$

$\therefore$ the 3[rd] row of Jacobian :

$$\frac{\partial p_z}{\partial \theta_1} = J_{31} = 0 \qquad \frac{\partial p_z}{\partial \theta_4} = J_{34} = [c_{234}a_4]$$

$$\frac{\partial p_z}{\partial \theta_2} = J_{32} = [c_{234}a_4 + c_{23}a_3 + c_2a_2] \qquad \frac{\partial p_z}{\partial \theta_5} = J_{35} = 0$$

$$\frac{\partial p_z}{\partial \theta_3} = J_{33} = [c_{234}a_4 + c_{23}a_3] \qquad \frac{\partial p_z}{\partial \theta_6} = J_{36} = 0$$

Calculation of the Jacobian

  - δx, δy, δz : rotation in T
  - But, no unique eq. that describes the rotations about the 3 axes. So, there is no single eq. for differential rotations about the 3 axes. → different calculation needed
- Simple Formula [Paul] :
  - The Jacobian w.r.t. T_6 (the last frame) is simpler to compute than that w.r.t. the first frame. [Paul]

$$\left[{}^{T_6}D \right] = \left[{}^{T_6}J \right] \left[D_\theta \right]$$

  - → for the same joint differential motions, premultiplied with the Jacobian w.r.t. the last frame, the hand differential motions w.r.t. the last frame is obtained.

Calculation of the Jacobian

- Simple Formula [Paul]

$$\begin{bmatrix} {}^{T_6}dx \\ {}^{T_6}dy \\ {}^{T_6}dz \\ {}^{T_6}\delta x \\ {}^{T_6}\delta y \\ {}^{T_6}\delta z \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} {}^{T_6}J_{11} \\ {}^{T_6}J_{21} \\ {}^{T_6}J_{31} \\ {}^{T_6}J_{41} \\ {}^{T_6}J_{51} \\ {}^{T_6}J_{61} \end{matrix}} & \boxed{\begin{matrix} {}^{T_6}J_{12} \\ {}^{T_6}J_{22} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix}} & \begin{matrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \end{matrix} & \boxed{\begin{matrix} {}^{T_6}J_{16} \\ {}^{T_6}J_{26} \\ {}^{T_6}J_{36} \\ {}^{T_6}J_{46} \\ {}^{T_6}J_{56} \\ {}^{T_6}J_{66} \end{matrix}} \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \\ d\theta_4 \\ d\theta_5 \\ d\theta_6 \end{bmatrix}$$

Use ${}^{i-1}T_6$ for the i^{th} column of ${}^{T_6}J$ $({}^{T_6}J_{ki})$

1st column: $\boxed{{}^0T_6 = A_1A_2A_3A_4A_5A_6}$

2nd column: $\boxed{{}^1T_6 = A_2A_3A_4A_5A_6}$

3rd column: $\boxed{{}^2T_6 = A_3A_4A_5A_6}$

4th column: ${}^3T_6 = A_4A_5A_6$

5th column: ${}^4T_6 = A_5A_6$

6th column: ${}^5T_6 = A_6$

- {n,o,a,p} matrix represented by any combination of $A_1 \sim A_6$ is used to calculate the Jacobian. (e.g., ${}^0T_6 \sim {}^5T_6$)
- i^{th} revolute joint : ${}^{T_6}J_{1i} = (-n_xp_y + n_yp_x)$ ${}^{T_6}J_{2i} = (-o_xp_y + o_yp_x)$ ${}^{T_6}J_{3i} = (-a_xp_y + a_yp_x)$
  ${}^{T_6}J_{4i} = n_z$ ${}^{T_6}J_{5i} = o_z$ ${}^{T_6}J_{6i} = a_z$
- i^{th} prismatic joint : ${}^{T_6}J_{1i} = n_z$ ${}^{T_6}J_{2i} = o_z$ ${}^{T_6}J_{3i} = a_z$
  ${}^{T_6}J_{4i} = 0$ ${}^{T_6}J_{5i} = 0$ ${}^{T_6}J_{6i} = 0$

Calculation of the Jacobian

- E 3.10: Find the T6J11 and T6J41 elements of the Jacobian for the simple revolute robot

$$^{R}T_H = A_1A_2A_3A_4A_5A_6$$

$$= \begin{bmatrix} c_1(c_{234}c_5c_6 - s_{234}s_6) & c_1(-c_{234}c_5c_6 - s_{234}s_6) & c_1(c_{234}s_5) & c_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \\ -s_1s_5c_6 & +s_1s_5s_6 & +s_1c_5 & \\ s_1(c_{234}c_5c_6 - s_{234}s_6) & s_1(-c_{234}c_5c_6 - s_{234}s_6) & s_1(c_{234}s_5) & s_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \\ +c_1s_5s_6 & -c_1s_5s_6 & -c_1c_5 & \\ s_{234}c_5c_6 + c_{234}s_6 & -s_{234}c_5c_6 + c_{234}c_6 & s_{234}s_5 & s_{234}a_4 + s_{23}a_3 + s_2a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For the ith revolute joint, by eq(3.25)

$$\boxed{\begin{aligned} {}^{T_6}J_{1i} &= (-n_xp_y + n_yp_x) \\ {}^{T_6}J_{4i} &= n_z \end{aligned}} \quad \begin{aligned} {}^{T_6}J_{2i} &= (-o_xp_y + o_yp_x) \\ {}^{T_6}J_{5i} &= o_z \end{aligned} \quad \begin{aligned} {}^{T_6}J_{3i} &= (-a_xp_y + a_yp_x) \\ {}^{T_6}J_{6i} &= a_z \end{aligned}$$

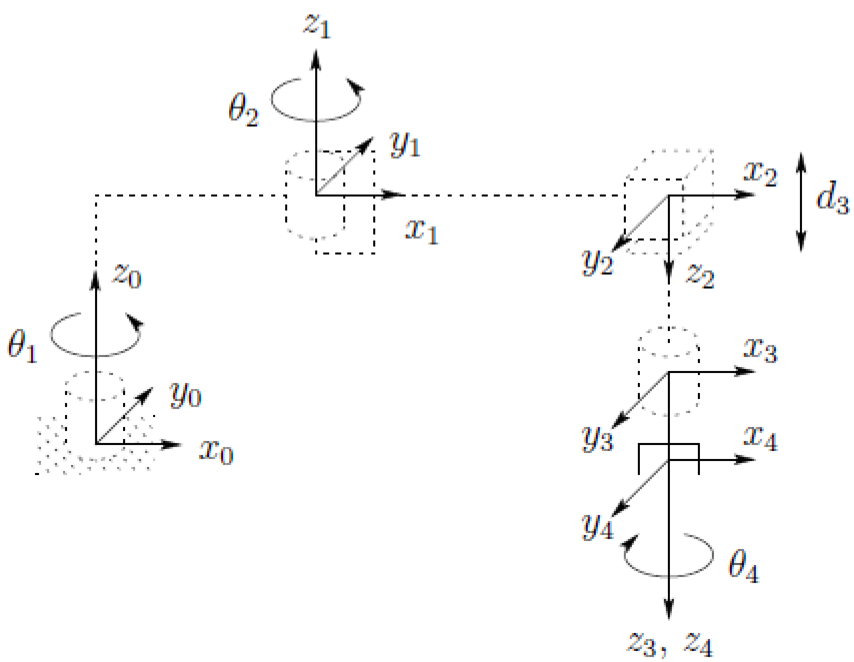
Substituting $i = 1$,

$$\begin{aligned} {}^{T_6}J_{11} = (-n_xp_y + n_yp_x) &= -[c_1(c_{234}c_5c_6 - s_{234}s_6) - s_1s_5c_6] \times [s_1(c_{234}a_4 + c_{23}a_3 + c_2a_2)] \\ &\quad + [s_1(c_{234}c_5c_6 - s_{234}s_6) + c_1s_5s_6] \times [c_1(c_{234}a_4 + c_{23}a_3 + c_2a_2)] \\ &= s_5c_6(c_{234}a_4 + c_{23}a_3 + c_2a_2) \end{aligned}$$

$${}^{T_6}J_{41} = n_z = s_{234}c_5c_6 + c_{234}s_6$$

Calculation of the Jacobian

SCARA Manipulator



Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	$\star$
2	a_2	180	0	$\star$
3	0	0	$\star$	0
4	0	0	d_4	$\star$

* joint variable

Jacobian of SCARA Manipulator

$$
\begin{aligned}
A_1 &= \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
A_2 &= \begin{bmatrix} c_2 & s_2 & 0 & a_2 c_2 \\ s_2 & -c_2 & 0 & a_2 s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
A_3 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
A_4 &= \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Jacobian of SCARA Manipulator

$$A_3A_4 = \begin{bmatrix} C_4 & -S_4 & 0 & 0 \\ S_4 & C_4 & 0 & 0 \\ 0 & 0 & 1 & d_3 + d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2A_3A_4 = \begin{bmatrix} C_2C_4 + S_2S_4 & S_2C_4 - C_2S_4 & 0 & a_2C_2 \\ S_2C_4 - C_2S_4 & -C_2C_4 - S_2S_4 & 0 & a_2S_2 \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1A_2A_3A_4 = \begin{bmatrix} C_{12}C_4 + S_{12}S_4 & S_{12}C_4 - C_{12}S_4 & 0 & a_1C_1 + a_2C_{12} \\ S_{12}C_4 - C_{12}S_4 & -C_{12}C_4 - S_{12}S_4 & 0 & a_1S_1 + a_2S_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Jacobian of SCARA Manipulator

- i^{th} revolute joint : ${}^{T_6}J_{1i} = (-n_x p_y + n_y p_x)$ ${}^{T_6}J_{2i} = (-o_x p_y + o_y p_x)$ ${}^{T_6}J_{3i} = (-a_x p_y + a_y p_x)$
  ${}^{T_6}J_{4i} = n_z$ ${}^{T_6}J_{5i} = o_z$ ${}^{T_6}J_{6i} = a_z$
- i^{th} prismatic joint : ${}^{T_6}J_{1i} = n_z$ ${}^{T_6}J_{2i} = o_z$ ${}^{T_6}J_{3i} = a_z$
  ${}^{T_6}J_{4i} = 0$ ${}^{T_6}J_{5i} = 0$ ${}^{T_6}J_{6i} = 0$

Since the first joint is revolute: (Consider $A_1A_2A_3A_4$)

$$J_{11} = -(C_{12}C_4 + S_{12}S_4)(a_1S_1 + a_2S_{12}) + (a_1C_1 + a_2C_{12})(S_{12}C_4 - C_{12}S_4)$$

$$J_{11} = -C(\theta_1 + \theta_2 - \theta_4)(a_1S_1 + a_2S_{12}) + S(\theta_1 + \theta_2 - \theta_4)(a_1C_1 + a_2C_{12})$$

$$J_{11} = a_1\left[S(\theta_1 + \theta_2 - \theta_4)C_1 - S_1C(\theta_1 + \theta_2 - \theta_4)\right] + a_2\left[S(\theta_1 + \theta_2 - \theta_4)C_{12} - S_{12}C(\theta_1 + \theta_2 - \theta_4)\right]$$

$$J_{11} = a_1S(\theta_2 - \theta_4) - a_2S_4$$

$$J_{21} = -S(\theta_1 + \theta_2 - \theta_4)(a_1S_1 + a_2S_{12}) - C(\theta_1 + \theta_2 - \theta_4)(a_1C_1 + a_2C_{12})$$

$$J_{21} = -a_1\left[S(\theta_1 + \theta_2 - \theta_4)S_1 + C_1C(\theta_1 + \theta_2 - \theta_4)\right] - a_2\left[S(\theta_1 + \theta_2 - \theta_4)S_{12} + C_{12}C(\theta_1 + \theta_2 - \theta_4)\right]$$

$$J_{21} = -a_1C(\theta_2 - \theta_4) - a_2C_4$$

Jacobian of SCARA Manipulator

- i^{th} revolute joint : ${}^{T_6}J_{1i} = (-n_x p_y + n_y p_x)$ ${}^{T_6}J_{2i} = (-o_x p_y + o_y p_x)$ ${}^{T_6}J_{3i} = (-a_x p_y + a_y p_x)$
  ${}^{T_6}J_{4i} = n_z$ ${}^{T_6}J_{5i} = o_z$ ${}^{T_6}J_{6i} = a_z$
- i^{th} prismatic joint : ${}^{T_6}J_{1i} = n_z$ ${}^{T_6}J_{2i} = o_z$ ${}^{T_6}J_{3i} = a_z$
  ${}^{T_6}J_{4i} = 0$ ${}^{T_6}J_{5i} = 0$ ${}^{T_6}J_{6i} = 0$

$$J_{31} = 0$$

$$J_{41} = 0$$

$$J_{51} = 0$$

$$J_{61} = -1$$

Jacobian of SCARA Manipulator

- i^{th} revolute joint : ${}^{T_6}J_{1i} = (-n_x p_y + n_y p_x)$ ${}^{T_6}J_{2i} = (-o_x p_y + o_y p_x)$ ${}^{T_6}J_{3i} = (-a_x p_y + a_y p_x)$
  ${}^{T_6}J_{4i} = n_z$ ${}^{T_6}J_{5i} = o_z$ ${}^{T_6}J_{6i} = a_z$
- i^{th} prismatic joint : ${}^{T_6}J_{1i} = n_z$ ${}^{T_6}J_{2i} = o_z$ ${}^{T_6}J_{3i} = a_z$
  ${}^{T_6}J_{4i} = 0$ ${}^{T_6}J_{5i} = 0$ ${}^{T_6}J_{6i} = 0$

(Consider $A_2A_3A_4$)

$$J_{12} = -a_2 S_4$$

$$J_{22} = -a_2 C_4$$

$$J_{32} = 0$$

$$J_{42} = 0$$

$$J_{52} = 0$$

$$J_{62} = -1$$

Jacobian of SCARA Manipulator

- i^{th} revolute joint : ${}^{T_6}J_{1i} = (-n_x p_y + n_y p_x)$ ${}^{T_6}J_{2i} = (-o_x p_y + o_y p_x)$ ${}^{T_6}J_{3i} = (-a_x p_y + a_y p_x)$
  ${}^{T_6}J_{4i} = n_z$ ${}^{T_6}J_{5i} = o_z$ ${}^{T_6}J_{6i} = a_z$
- i^{th} prismatic joint : ${}^{T_6}J_{1i} = n_z$ ${}^{T_6}J_{2i} = o_z$ ${}^{T_6}J_{3i} = a_z$
  ${}^{T_6}J_{4i} = 0$ ${}^{T_6}J_{5i} = 0$ ${}^{T_6}J_{6i} = 0$

(Consider A_3A_4)

$$J_{13} = 0$$
$$J_{23} = 0$$
$$J_{33} = 1$$
$$J_{43} = 0$$
$$J_{53} = 0$$
$$J_{63} = 0$$

Jacobian of SCARA Manipulator

- i^{th} revolute joint : ${}^{T_6}J_{1i} = (-n_x p_y + n_y p_x)$ ${}^{T_6}J_{2i} = (-o_x p_y + o_y p_x)$ ${}^{T_6}J_{3i} = (-a_x p_y + a_y p_x)$
  ${}^{T_6}J_{4i} = n_z$ ${}^{T_6}J_{5i} = o_z$ ${}^{T_6}J_{6i} = a_z$
- i^{th} prismatic joint : ${}^{T_6}J_{1i} = n_z$ ${}^{T_6}J_{2i} = o_z$ ${}^{T_6}J_{3i} = a_z$
  ${}^{T_6}J_{4i} = 0$ ${}^{T_6}J_{5i} = 0$ ${}^{T_6}J_{6i} = 0$

(Consider A_4)

$$J_{14} = 0$$
$$J_{24} = 0$$
$$J_{34} = 0$$
$$J_{44} = 0$$
$$J_{54} = 0$$
$$J_{64} = 1$$

Jacobian of SCARA Manipulator

Hence

$$\begin{bmatrix} {}^{T_4}dx \\ {}^{T_4}dy \\ {}^{T_4}dz \\ {}^{T_4}\delta x \\ {}^{T_4}\delta y \\ {}^{T_4}\delta z \end{bmatrix} = \begin{bmatrix} a_1 S(\theta_2 - \theta_4) - a_2 S_4 & -a_2 S_4 & 0 & 0 \\ -a_1 C(\theta_2 - \theta_4) - a_2 C_4 & -a_2 C_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ dd_3 \\ d\theta_4 \end{bmatrix}$$

Relationship between the Jacobians w.r.t base and the hand frame

$$\begin{bmatrix} \mathbf{J_d} \\ \hline \mathbf{J_\delta} \end{bmatrix} = \left[\begin{array}{c|c} {}^0\mathbf{R}_n & \mathbf{0} \\ \hline \mathbf{0} & {}^0\mathbf{R}_n \end{array}\right] \begin{bmatrix} {}^{T_n}\mathbf{J_d} \\ \hline {}^{T_n}\mathbf{J_\delta} \end{bmatrix}$$

How to relate the Jacobian and the Differential Operator

- Jacobian vs. Differential Operator
  - differential joint motions → J → D={dx, dy, dz, δx, δy, δz} → substituted in Δ → differential transformation (dT=ΔT) → *update position/orientation of robot's hand*

$$[D]=[J][D_\theta] \qquad \begin{bmatrix} & & \\ & \text{Robot} & \\ & \text{Jacobian Matrix} & \\ & & \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \\ d\theta_4 \\ d\theta_5 \\ d\theta_6 \end{bmatrix} = \begin{bmatrix} dx \\ dy \\ dz \\ \delta x \\ \delta y \\ \delta z \end{bmatrix} \qquad \Delta = \begin{bmatrix} 0 & -\delta z & \delta y & dx \\ \delta z & 0 & -\delta x & dy \\ -\delta y & \delta x & 0 & dz \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  - alternatively, differential joint motions → ${}^{T6}J$ → ${}^{T6}D$={${}^{T6}dx$, ${}^{T6}dy$, ${}^{T6}dz$, ${}^{T6}\delta x$, ${}^{T6}\delta y$, ${}^{T6}\delta z$} → substituted in ${}^{T}\Delta$ → differential transformation (dT=T${}^{T}\Delta$) → *update position/orientation of robot's hand*

How to relate the Jacobian and the Differential Operator

- Example 3.11 The hand frame of a 5-DOF robot, its numerical Jacobian for this instance, and a set of differential motions are given. The robot has a 2RP2R configuration. Find the new location of the hand after the differential motion.

$$T_6 = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ ds_3 \\ d\theta_4 \\ d\theta_5 \end{bmatrix} = \begin{bmatrix} 0.1 \\ -0.1 \\ 0.05 \\ 0.1 \\ 0 \end{bmatrix}$$

- Sol.

$$[D] = [J][D_\theta] \quad \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1 \\ -0.1 \\ 0.05 \\ 0.1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.3 \\ -0.15 \\ -0.4 \\ 0 \\ -0.1 \end{bmatrix} \quad \rightarrow \quad \Delta = \begin{bmatrix} 0 & 0 & -0.1 & 0.3 \\ 0 & 0 & 0 & -0.15 \\ 0.1 & 0 & 0 & -0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[dT_6] = [\Delta][T_6] = \begin{bmatrix} 0 & 0 & -0.1 & 0.3 \\ 0 & 0 & 0 & -0.15 \\ 0.1 & 0 & 0 & -0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -0.1 & 0 & 0.1 \\ 0 & 0 & 0 & -0.15 \\ 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{6,new} = dT_6 + T_{6,old} = \begin{bmatrix} 0 & -0.1 & 0 & 0.1 \\ 0 & 0 & 0 & -0.15 \\ 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -0.1 & 0 & 5.1 \\ 0 & 0 & -1 & 2.85 \\ 0.1 & 1 & 0 & 2.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Jacobian

Inverse Jacobian

- To compute the differential velocities at the joints of the robot for a desired hand differential velocities.

$$\left.\begin{array}{r}[D]=[J][D_\theta] \\ [J^{-1}][D]=[J^{-1}][J][D_\theta]\end{array}\right\}\rightarrow [D_\theta]=[J^{-1}][D]$$

$$\left.\begin{array}{r}[{}^{T_6}D]=[{}^{T_6}J][D_\theta] \\ [{}^{T_6}J^{-1}][D]=[{}^{T_6}J^{-1}][{}^{T_6}J][D_\theta]\end{array}\right\}\rightarrow [D_\theta]=[{}^{T_6}J^{-1}][{}^{T_6}D]$$

- ⇨ The robot follows a <u>desired path (inverse kinematics)</u> on a flat plane with a <u>constant speed (inverse Jacobian)</u>.

Inverse Jacobian

- Problem
  - To compute the inverse Jacobian <u>in real-time</u> ~ difficult (time consuming and computationally intensive)
- Sol. (2-ways)
  - 1) find the symbolic inverse of the Jacobian → substitute the values into inv(J) to compute the velocities
  - 2) substitute the numbers in Jacobian → invert the numerical matrix by Gaussian elimination or LU decomposition
- Simpler Sol.
  - Use the inverse kinematic eq.'s to compute joint velocities
  - $d\theta_{i=1\sim6}$ = differentiate{inverse kinematics} = J(dn,do,da,dp)

Inverse Jacobian

- Simpler Sol. (inverse kinematics → inverse Jacobian)

$$p_x s_1 - p_y c_1 = 0 \quad \rightarrow \quad \theta_1 = \tan^{-1}\left(\frac{p_y}{p_x}\right) \text{ and } \theta_1 = \theta_1 + 180^\circ$$

$$p_x s_1 = p_y c_1 \quad \rightarrow \quad \text{differentiate both sides for } d\theta_1$$

$$dp_x s_1 + p_x c_1 d\theta_1 = dp_y c_1 - p_y s_1 d\theta_1 \quad \rightarrow \quad d\theta_1 (p_x c_1 + p_y s_1) = -dp_x s_1 + dp_y c_1$$

$$\therefore \quad d\theta_1 = \frac{-dp_x s_1 + dp_y c_1}{(p_x c_1 + p_y s_1)}$$

$$s_{234}(c_1 a_x + s_1 a_y) = c_{234} a_z \quad \rightarrow \quad \theta_{234} = \tan^{-1}\left(\frac{a_z}{c_1 a_x + s_1 a_y}\right) \text{ and } \theta_{234} = \theta_{234} + 180^\circ$$

$$\rightarrow \quad \text{differentiate both sides}$$

$$c_{234}(d\theta_2 + d\theta_3 + d\theta_4)(c_1 a_x + s_1 a_y) + s_{234}\left[-s_1 a_x d\theta_1 + c_1 da_x + c_1 a_y d\theta_1 + s_1 da_y\right]$$

$$= s_{234}(d\theta_2 + d\theta_3 + d\theta_4) a_z + c_{234} da_z$$

$$\therefore \quad (d\theta_2 + d\theta_3 + d\theta_4) = \frac{s_{234}\left[-s_1 a_x d\theta_1 + c_1 da_x + c_1 a_y d\theta_1 + s_1 da_y\right] + c_{234} da_z}{c_{234}(c_1 a_x + s_1 a_y) + s_{234} a_z}$$

Inverse Jacobian

- Simpler Sol. (inverse kinematics → inverse Jacobian)

$$2a_2a_3c_3 = (p_xc_1 + p_ys_1 - c_{234}a_4)^2 + (p_z - s_{234}a_4)^2 - a_2^2 - a_3^2 \quad \rightarrow \quad \theta_3 = \tan^{-1}\left(\frac{s_3}{c_3}\right)$$
$$s_3 = \pm\sqrt{1-c_3^2}$$

→ differentiate both sides

$$-2a_2a_3s_3 d\theta_3 = 2(p_xc_1 + p_ys_1 - c_{234}a_4)[dp_xc_1 - p_xs_1d\theta_1 + dp_ys_1 + p_yc_1d\theta_1 + s_{234}a_4(d\theta_2 + d\theta_3 + d\theta_4)]$$
$$+ 2(p_z - s_{234}a_4)[dp_z - c_{234}a_4(d\theta_2 + d\theta_3 + d\theta_4)]$$

- eq(2.70) : $s_2\left[(c_3a_3 + a_2)^2 + s_3^2a_3^2\right] = (c_3a_3 + a_2)(p_z - s_{234}a_4) - s_3a_3(p_xc_1 + p_ys_1 - c_{234}a_4)$

→ differentiate both sides

$$c_2 d\theta_2\left[(c_3a_3 + a_2)^2 + s_3^2a_3^2\right] + s_2\left[2(c_3a_3 + a_2)(-s_3a_3d\theta_3) + 2s_3c_3a_3^2d\theta_3\right]$$
$$= (-s_3a_3d\theta_3)(p_z - s_{234}a_4) + (c_3a_3 + a_2)[dp_z - c_{234}a_4(d\theta_2 + d\theta_3 + d\theta_4)]$$
$$- c_3a_3d\theta_3(p_xc_1 + p_ys_1 - c_{234}a_4) - s_3a_3[dp_xc_1 - p_xs_1d\theta_1 + dp_ys_1 + p_yc_1d\theta_1 + s_{234}a_4(d\theta_2 + d\theta_3 + d\theta_4)]$$

$$\therefore\ d\theta_4 = \frac{s_{234}[-s_1a_xd\theta_1 + c_1da_x + c_1a_yd\theta_1 + s_1da_y] + c_{234}da_z}{c_{234}(c_1a_x + s_1a_y) + s_{234}a_z} - d\theta_2 - d\theta_3$$

Inverse Jacobian

- Simpler Sol. (inverse kinematics → inverse Jacobian)

$$c_5 = -c_1 a_y + s_1 a_x$$

→ differentiate both sides

$$-s_5 d\theta_5 = s_1 a_y d\theta_1 - c_1 da_y + c_1 a_x d\theta_1 + s_1 da_x$$

$$s_6 = -s_{234}\left(c_1 n_x + s_1 n_y\right) + c_{234} n_z$$

→ differentiate both sides

$$c_6 d\theta_6 = -c_{234}\left(c_1 n_x + s_1 n_y\right)\left(d\theta_2 + d\theta_3 + d\theta_4\right)$$
$$-s_{234}\left(-s_1 n_x d\theta_1 + c_1 dn_x + c_1 n_y d\theta_1 + s_1 dn_y\right)$$
$$-s_{234} n_z \left(d\theta_2 + d\theta_3 + d\theta_4\right) + c_{234} dn_z$$

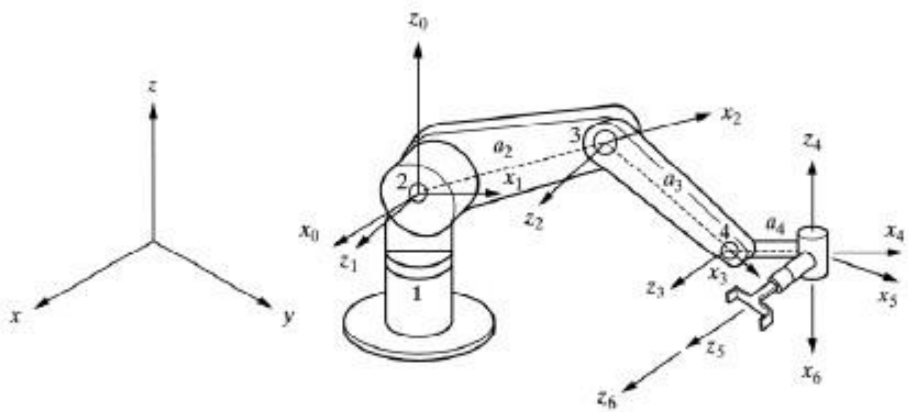
Inverse Jacobian

Example 3.16

The revolute robot of Example 2.25 is in the following configuration. Calculate the angular velocity of the first joint for the given values such that the hand frame will have the following linear and angular velocities:

$$dx/dt = 1\ \text{in/sec} \quad dy/dt = -2\ \text{in/sec} \quad \delta x/dt = 0.1\ \text{rad/sec}$$

$$\theta_1 = 0°, \quad \theta_2 = 90°, \quad \theta_3 = 0°, \quad \theta_4 = 90°, \quad \theta_5 = 0°, \quad \theta_6 = 45°$$
$$a_2 = 15'', \quad a_3 = 15'', \quad a_4 = 5''$$



D–H parameters				
#	Θ	d	a	α
1	Θ_1	0	0	90
2	Θ_2	0	a_2	0
3	Θ_3	0	a_3	0
4	Θ_4	0	a_4	−90
5	Θ_5	0	0	90
6	Θ_6	0	0	0

Inverse Jacobian

$$^{R}T_{H} = A_1A_2A_3A_4A_5A_6$$

$$= \begin{bmatrix} c_1(c_{234}c_5c_6 - s_{234}s_6) & c_1(-c_{234}c_5c_6 - s_{234}s_6) & c_1(c_{234}s_5) & c_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \\ -s_1s_5c_6 & +s_1s_5s_6 & +s_1c_5 & \\ s_1(c_{234}c_5c_6 - s_{234}s_6) & s_1(-c_{234}c_5c_6 - s_{234}s_6) & s_1(c_{234}s_5) & s_1(c_{234}a_4 + c_{23}a_3 + c_2a_2) \\ +c_1s_5s_6 & -c_1s_5s_6 & -c_1c_5 & \\ s_{234}c_5c_6 + c_{234}s_6 & -s_{234}c_5c_6 + c_{234}c_6 & s_{234}s_5 & s_{234}a_4 + s_{23}a_3 + s_2a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{R}T_{H} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.707 & 0.707 & 0 & -5 \\ 0 & 0 & -1 & 0 \\ -0.707 & -0.707 & 0 & 30 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 0 & -\delta z & \delta y & dx \\ \delta z & 0 & -\delta x & dy \\ -\delta y & \delta x & 0 & dz \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -0.1 & -2 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [dT] = [\Delta][T] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0.707 & 0.707 & 0 & -5 \\ 0 & 0 & -0.1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad \frac{d\theta_1}{dt} = \frac{-dp_x s_1 + dp_y c_1}{(p_x c_1 + p_y s_1)} = \frac{-1(0) - 5(1)}{-5(1) + 0(0)} = 1 \ \text{(rad/sec)}$$

Thank you!