INDUSTRIAL AUTOMATION & ROBOTICS TECHNOLOGY

Inverse Kinematics + Differential Motions and Velocities

M Umer khan

Department of Mechatronics Engineering

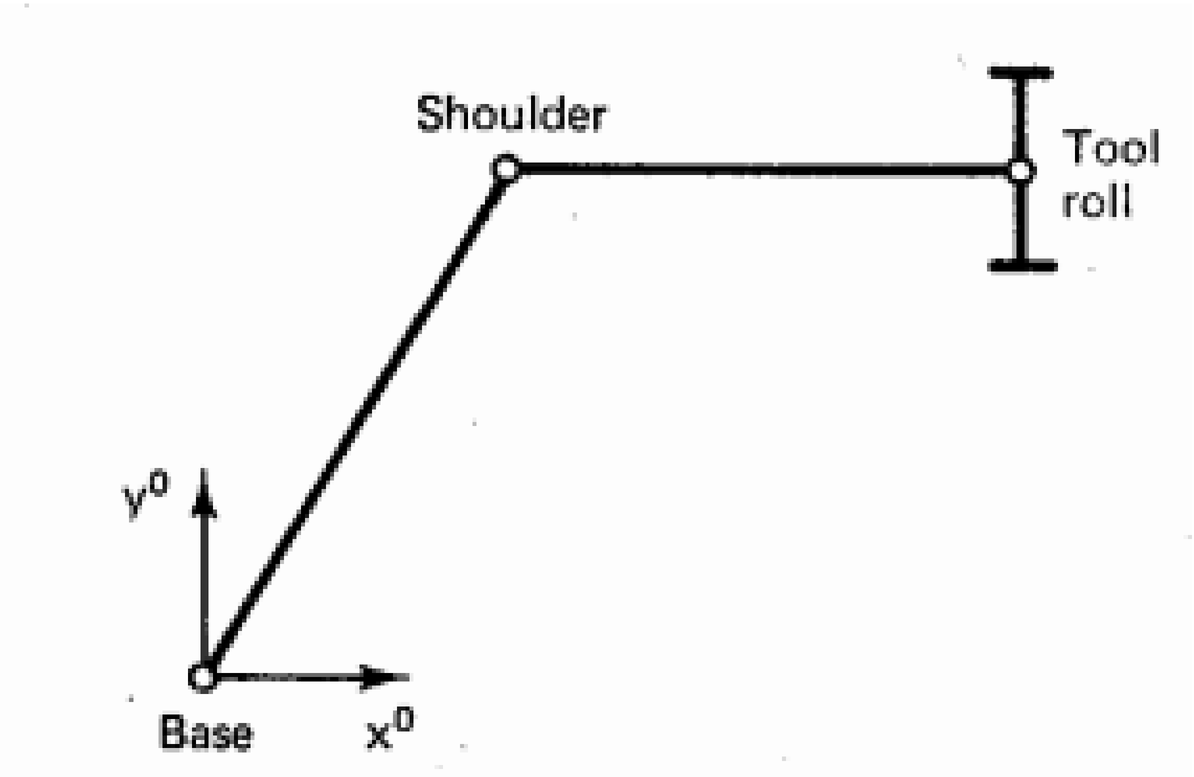
ATILIM University

UMER.KHAN@ATILIM.EDU.TR

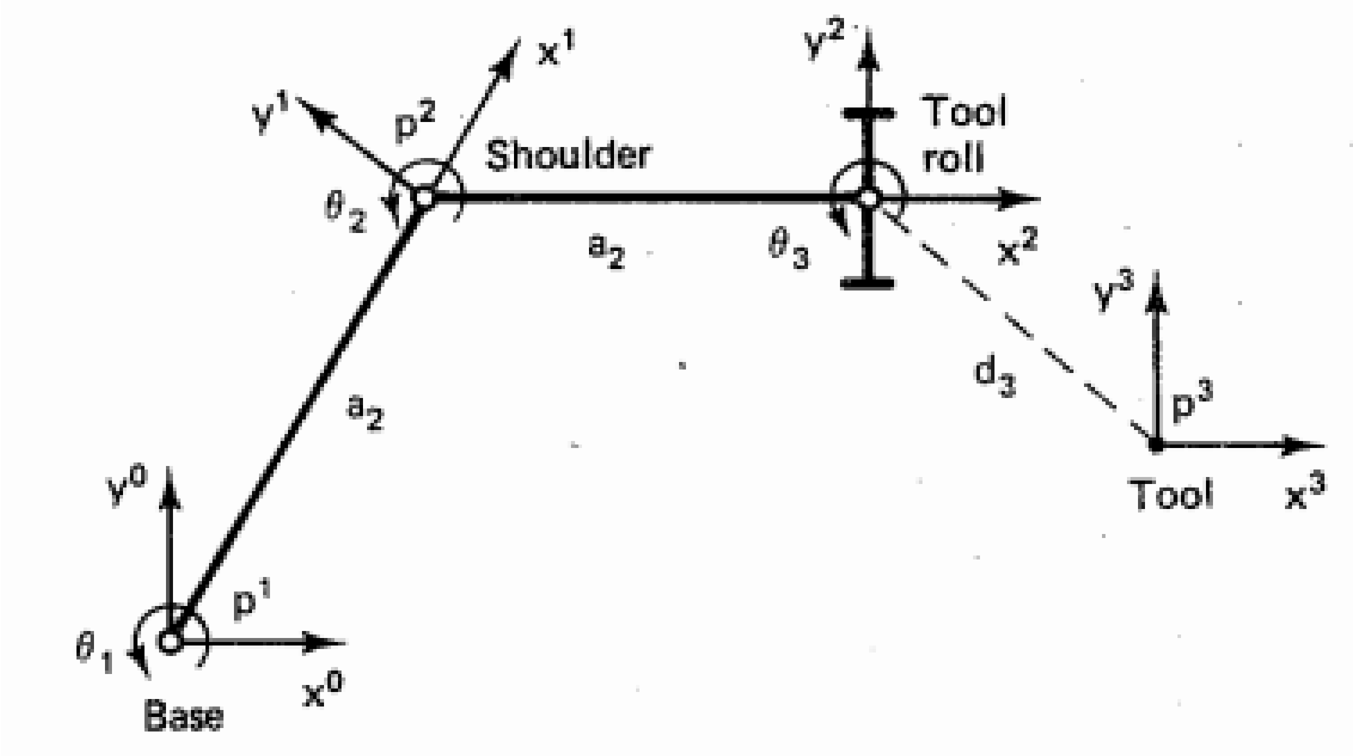
Materials used

- Chapter 4 Introduction to Robotics, John J. Craig
- Chapter 3, Fundamentals of Robotics, Analysis and Control, Robert Schilling
- Chapter 3, Introduction to Robotics, Saeed B. Niku

IK of a three-axis articulated robot



IK of a three-axis articulated robot



IK of a three-axis articulated robot

DH parameters

Axis	θ	d	a	α	Home
1	q_1	0	a_1	0	$\pi/3$
2	q_2	0	a_2	0	$-\pi/3$
3	q_3	d_3	0	0	0

IK of a three-axis articulated robot

Arm matrix

$$
\begin{aligned}
T_{\text{base}}^{\text{tool}} &= T_0^1 T_1^2 T_2^3 \\
&= \begin{bmatrix} C_1 & -S_1 & 0 & a_1C_1 \\ S_1 & C_1 & 0 & a_1S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} C_2 & -S_2 & 0 & a_2C_2 \\ S_2 & C_2 & 0 & a_2S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} C_3 & -S_3 & 0 & 0 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} C_{12} & -S_{12} & 0 & a_1C_1 + a_2C_{12} \\ S_{12} & C_{12} & 0 & a_1S_1 + a_2S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} C_3 & -S_3 & 0 & 0 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} C_{123} & -S_{123} & 0 & a_1C_1 + a_2C_{12} \\ S_{123} & C_{123} & 0 & a_1S_1 + a_2S_{12} \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

IK of a three-axis articulated robot

Tool configuration vector

$$w(q) = \begin{bmatrix} a_1 C_1 + a_2 C_{12} \\ a_1 S_1 + a_2 S_{12} \\ d_3 \\ \hdashline 0 \\ 0 \\ \exp(q_3/\pi) \end{bmatrix}$$

IK of a three-axis articulated robot

Shoulder joint q_2

$$\begin{aligned} w_1^2 + w_2^2 &= (a_1 C_1 + a_2 C_{12})^2 + (a_1 S_1 + a_2 S_{12})^2 \\ &= a_1^2 C_1^2 + 2a_1 a_2 C_1 C_{12} + a_2^2 C_{12}^2 + a_1^2 S_1^2 + 2a_1 a_2 S_1 S_{12} + a_2^2 S_{12}^2 \\ &= a_1^2 + 2a_1 a_2 (C_{12} C_1 + S_{12} S_1) + a_2^2 \\ &= a_1^2 + 2a_1 a_2 C_2 + a_2^2 \end{aligned}$$

Isolate C_2

$$q_2 = \pm \arccos \frac{w_1^2 + w_2^2 - a_1^2 - a_2^2}{2a_1 a_2}$$

IK of a three-axis articulated robot

Base joint q_1

$$(a_1 + a_2 C_2)C_1 - (a_2 S_2)S_1 = w_1$$

$$(a_2 S_2)C_1 + (a_1 + a_2 C_2)S_1 = w_2,$$

Solve simultaneously

$$C_1 = \frac{(a_1 + a_2 C_2)w_1 + a_2 S_2 w_2}{(a_1 + a_2 C_2)^2 + (a_2 S_2)^2}$$

$$S_1 = \frac{(a_1 + a_2 C_2)w_2 - a_2 S_2 w_1}{(a_1 + a_2 C_2)^2 + (a_2 S_2)^2}$$

Complete solution

$$q_1 = \operatorname{atan2}\left[(a_1 + a_2 C_2)w_2 - a_2 S_2 w_1, (a_1 + a_2 C_2)w_1 + a_2 S_2 w_2\right]$$

IK of a three-axis articulated robot

Tool roll joint q_3

$$q_3 = \pi \ln w_6$$

IK of a three-axis articulated robot

Complete solution

$$q_2 = \pm\arccos \frac{w_1^2 + w_2^2 - a_1^2 - a_2^2}{2a_1 a_2}$$

$$q_1 = \operatorname{atan2}\left[(a_1 - a_2 C_2)w_1 + a_2 S_2 w_2,\ (a_1 + a_2 C_2)w_2 - a_2 S_2 w_1\right]$$

$$q_3 = \pi \ln w_6$$

IK of a four-axis SCARA robot (ADEPT ONE)

» Arm matrix

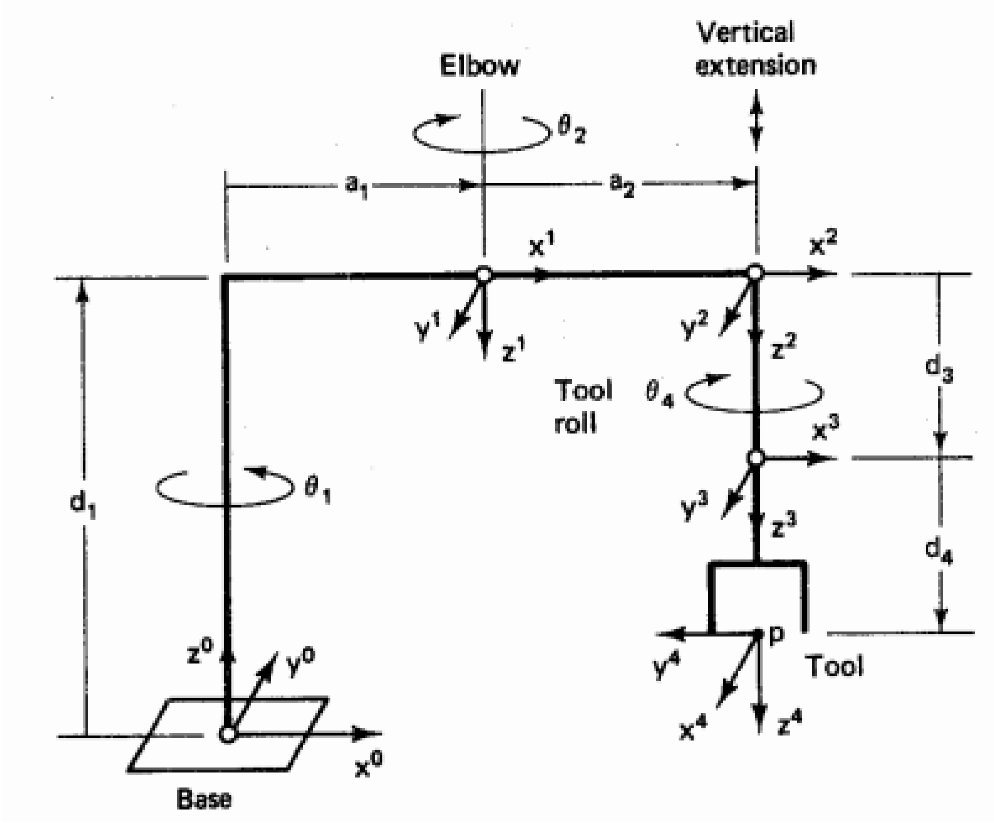
$$T_{\text{base}}^{\text{tool}} = \left[\begin{array}{ccc:c} C_{1-2-4} & S_{1-2-4} & 0 & a_1C_1 + a_2C_{1-2} \\ S_{1-2-4} & -C_{1-2-4} & 0 & a_1S_1 + a_2S_{1-2} \\ 0 & 0 & -1 & d_1 - q_3 - d_4 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right]$$

» Tool configuration vector

$$w(q) = \left[\begin{array}{c} a_1C_1 + a_2C_{1-2} \\ a_1S_1 + a_2S_{1-2} \\ d_1 - q_3 - d_4 \\ \hdashline 0 \\ 0 \\ -\exp(q_4/\pi) \end{array}\right]$$

IK of a four-axis SCARA robot (ADEPT ONE)

» Link coordinate diagram



IK of a four-axis SCARA robot (ADEPT ONE)

» Elbow joint q_2

$$w_1^2 + w_2^2 = a_1^2 + 2a_1a_2C_2 + a_2^2$$

» Two solutions are obtained as left-handed solution and right-handed solution

$$q_2 = \pm \arccos \frac{w_1^2 + w_2^2 - a_1^2 - a_2^2}{2a_1a_2}$$

IK of a four-axis SCARA robot (ADEPT ONE)

Base joint q_1

$$(a_1 + a_2 C_2) C_1 + (a_2 S_2) S_1 = w_1$$

$$(-a_2 S_2) C_1 + (a_1 + a_2 C_2) S_1 = w_2$$

» Use row operations to solve linear system

$$S_1 = \frac{a_2 S_2 w_1 + (a_1 + a_2 C_2) w_2}{(a_2 S_2)^2 + (a_1 + a_2 C_2)^2}$$

$$C_1 = \frac{(a_1 + a_2 C_2) w_1 - a_2 S_2 w_2}{(a_2 S_2)^2 + (a_1 + a_2 C_2)^2}$$

» Recover the base angle over the complete range $[-\pi, \pi]$

$$q_1 = \operatorname{atan2}\left[a_2 S_2 w_1 + (a_1 + a_2 C_2) w_2,\ (a_1 + a_2 C_2) w_1 - a_2 S_2 w_2\right]$$

IK of a four-axis SCARA robot (ADEPT ONE)

Vertical extension joint q_3

- » The prismatic joint variable is associated with sliding the tool up and down along the roll axis
- » In SCARA-type robot, the vertical component of the tool motion is uncoupled from the horizontal component.
- » Extracting the prismatic joint variable is simple

$$q_3 = d_1 - d_4 - w_3$$

IK of a four-axis SCARA robot (ADEPT ONE)

Tool roll joint q_4

» Inspection of the last component of tool configuration vector reveals that the tool roll joint can be easily recovered

$$q_4 = \pi \ln |w_6|$$

IK of a four-axis SCARA robot (ADEPT ONE)

Complete solution

$$q_2 = \pm \arccos \frac{w_1^2 + w_2^2 - a_1^2 - a_2^2}{2a_1 a_2}$$

$$q_1 = \operatorname{atan2}\left[a_2 S_2 w_1 + (a_1 + a_2 C_2) w_2,\ (a_1 + a_2 C_2) w_1 - a_2 S_2 w_2\right]$$

$$q_3 = d_1 - d_4 - w_3$$

$$q_4 = \pi \ln |w_6|$$

IK of PUMA 560

» Algebraic solution

$$
{}^0_6T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= {}^0_1T(\theta_1)\,{}^1_2T(\theta_2)\,{}^2_3T(\theta_3)\,{}^3_4T(\theta_4)\,{}^4_5T(\theta_5)\,{}^5_6T(\theta_6)
$$

» inverse of first frame

$$
[{}^0_1T(\theta_1)]^{-1}\,{}^0_6T = {}^1_2T(\theta_2)\,{}^2_3T(\theta_3)\,{}^3_4T(\theta_4)\,{}^4_5T(\theta_5)\,{}^5_6T(\theta_6).
$$

$$
\begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^1_6T,
$$

IK of PUMA 560

» Forward kinematics solution

$$
{}^{1}_{6}T = {}^{1}_{3}T\,{}^{3}_{6}T = \begin{bmatrix} {}^{1}r_{11} & {}^{1}r_{12} & {}^{1}r_{13} & {}^{1}p_x \\ {}^{1}r_{21} & {}^{1}r_{22} & {}^{1}r_{23} & {}^{1}p_y \\ {}^{1}r_{31} & {}^{1}r_{32} & {}^{1}r_{33} & {}^{1}p_z \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
\begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^{1}_{6}T,
$$

$$
\begin{aligned}
{}^{1}r_{11} &= c_{23}[c_4c_5c_6 - s_4s_6] - s_{23}s_5s_6, \\
{}^{1}r_{21} &= -s_4c_5c_6 - c_4s_6, \\
{}^{1}r_{31} &= -s_{23}[c_4c_5c_6 - s_4s_6] - c_{23}s_5c_6, \\
{}^{1}r_{12} &= -c_{23}[c_4c_5s_6 + s_4c_6] + s_{23}s_5s_6, \\
{}^{1}r_{22} &= s_4c_5s_6 - c_4c_6, \\
{}^{1}r_{32} &= s_{23}[c_4c_5s_6 + s_4c_6] + c_{23}s_5s_6, \\
{}^{1}r_{13} &= -c_{23}c_4s_5 - s_{23}c_5, \\
{}^{1}r_{23} &= s_4s_5, \\
{}^{1}r_{33} &= s_{23}c_4s_5 - c_{23}c_5, \\
{}^{1}p_x &= a_2c_2 + a_3c_{23} - d_4s_{23}, \\
{}^{1}p_y &= d_3, \\
{}^{1}p_x &= -a_3s_{23} - a_2s_2 - d_4c_{23}.
\end{aligned}
$$

IK of PUMA 560

» By comparing (2,4) on both sides

$$-s_1 p_x + c_1 p_y = d_3.$$

» To solve an equation of this form, we make the trigonometric substitutions

$$p_x = \rho \cos\phi,$$
$$p_y = \rho \sin\phi,$$

$$c_1 s_\phi - s_1 c_\phi = \frac{d_3}{\rho}.$$

$$\rho = \sqrt{p_x^2 + p_y^2},$$
$$\phi = \text{Atan2}(p_y, p_x).$$

IK of PUMA 560

» Using difference of angles

$$\sin(\phi - \theta_1) = \frac{d_3}{\rho}.$$

$$\cos(\phi - \theta_1) = \pm\sqrt{1 - \frac{d_3^2}{\rho^2}},$$

$$\phi - \theta_1 = \text{Atan2}\left(\frac{d_3}{\rho}, \pm\sqrt{1 - \frac{d_3^2}{\rho^2}}\right).$$

$$\theta_1 = \text{Atan2}(p_y, p_x) - \text{Atan2}\left(d_3, \pm\sqrt{p_x^2 + p_y^2 - d_3^2}\right).$$

IK of PUMA 560

» Equating (1,4)

$$c_1 p_x + s_1 p_y = a_3 c_{23} - d_4 s_{23} + a_2 c_2,$$

» Equating (3,4)

$$-s_1 p_x + c_1 p_y = d_3.$$

$$-p_x = a_3 s_{23} + d_4 c_{23} + a_2 s_2.$$

$$a_3 c_3 - d_4 s_3 = K,$$

$$K = \frac{p_x^2 + p_y^2 + p_x^2 - a_2^2 - a_3^2 - d_3^2 - d_4^2}{2a_2}.$$

IK of PUMA 560

$$\theta_3 = \mathrm{Atan2}(a_3, d_4) - \mathrm{Atan2}(K, \pm\sqrt{a_3^2 + d_4^2 - K^2}).$$

$$[{}^0_3T(\theta_2)]^{-1}\,{}^0_6T = {}^3_4T(\theta_4)\,{}^4_5T(\theta_5)\,{}^5_6T(\theta_6),$$

$$\begin{bmatrix} c_1c_{23} & s_1c_{23} & -s_{23} & -a_2c_3 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 \\ -s_1 & c_1 & 0 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^3_6T,$$

IK of PUMA 560

$$\theta_3 = \text{Atan2}(a_3, d_4) - \text{Atan2}(K, \pm\sqrt{a_3^2 + d_4^2 - K^2}).$$

$$[{}^0_3T(\theta_2)]^{-1}{}^0_6T = {}^3_4T(\theta_4){}^4_5T(\theta_5){}^5_6T(\theta_6),$$

$$\begin{bmatrix} c_1c_{23} & s_1c_{23} & -s_{23} & -a_2c_3 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 \\ -s_1 & c_1 & 0 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^3_6T,$$

IK of PUMA 560

» Equating (1,4) and (2,4) from both sides

$$c_1 c_{23} p_x + s_1 c_{23} p_y - s_{23} p_z - a_2 c_3 = a_3,$$

$$-c_1 s_{23} p_x - s_1 s_{23} p_y - c_{23} p_z + a_2 s_3 = d_4.$$

» Solve both equations simultaneously

$$s_{23} = \frac{(-a_3 - a_2 c_3) p_z + (c_1 p_x + s_1 p_y)(a_2 s_3 - d_4)}{p_z^2 + (c_1 p_x + s_1 p_y)^2},$$

$$c_{23} = \frac{(a_2 s_3 - d_4) p_z - (a_3 + a_2 c_3)(c_1 p_x + s_1 p_y)}{p_z^2 + (c_1 p_x + s_1 p_y)^2}.$$

IK of PUMA 560

» Denominators are positive and equal

$$\theta_{23} = \text{Atan2}[(-a_3 - a_2 c_3) p_z - (c_1 p_x + s_1 p_y)(d_4 - a_2 s_3),$$
$$(a_2 s_3 - d_4) p_z - (a_3 + a_2 c_3)(c_1 p_x + s_1 p_y)].$$

$$\theta_2 = \theta_{23} - \theta_3,$$

» Equating (1,3) and (3,3)

$$r_{13} c_1 c_{23} + r_{23} s_1 c_{23} - r_{33} s_{23} = -c_4 s_5,$$
$$-r_{13} s_1 + r_{23} c_1 = s_4 s_5.$$

IK of PUMA 560

» Denominators are positive and equal

$$\theta_{23} = \text{Atan2}[(-a_3 - a_2 c_3) p_z - (c_1 p_x + s_1 p_y)(d_4 - a_2 s_3),$$
$$(a_2 s_3 - d_4) p_z - (a_3 + a_2 c_3)(c_1 p_x + s_1 p_y)].$$

$$\theta_2 = \theta_{23} - \theta_3,$$

» Equating (1,3) and (3,3)

$$r_{13} c_1 c_{23} + r_{23} s_1 c_{23} - r_{33} s_{23} = -c_4 s_5,$$
$$-r_{13} s_1 + r_{23} c_1 = s_4 s_5.$$

IK of PUMA 560

» $s_5 \neq 0$

$$\theta_4 = \text{Atan2}(-r_{13}s_1 + r_{23}c_1, -r_{13}c_1c_{23} - r_{23}s_1c_{23} + r_{33}s_{23}).$$

$$[{}^0_4T(\theta_4)]^{-1}\ {}^0_6T = {}^4_5T(\theta_5){}^5_6T(\theta_6),$$

» Inverse of 0-4 matrix

$$\begin{bmatrix} c_1c_{23}c_4 + s_1s_4 & s_1c_{23}c_4 - c_1s_4 & -s_{23}c_4 & -a_2c_3c_4 + d_3s_4 - a_3c_4 \\ -c_1c_{23}s_4 + s_1c_4 & -s_1c_{23}s_4 - c_1c_4 & s_{23}s_4 & a_2c_3s_4 + d_3c_4 + a_3s_4 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

IK of PUMA 560

» Equating (1,3) and (3,3)

$$r_{13}(c_1 c_{23} c_4 + s_1 s_4) + r_{23}(s_1 c_{23} c_4 - c_1 s_4) - r_{33}(s_{23} c_4) = -s_5,$$

$$r_{13}(-c_1 s_{23}) + r_{23}(-s_1 s_{23}) + r_{33}(-c_{23}) = c_5.$$

$$\theta_5 = \text{Atan2}(s_5, c_5),$$

$$({}^0_5T)^{-1}\,{}^0_6T = {}^5_6T(\theta_6).$$

» Equating (3,1) and (1,1)

$$\theta_6 = \text{Atan2}(s_6, c_6),$$

IK of PUMA 560

$$s_6 = -r_{11}(c_1c_{23}s_4 - s_1c_4) - r_{21}(s_1c_{23}s_4 + c_1c_4) + r_{31}(s_{23}s_4),$$

$$c_6 = r_{11}[(c_1c_{23}c_4 + s_1s_4)c_5 - c_1s_{23}s_5] + r_{21}[(s_1c_{23}c_4 - c_1s_4)c_5 - s_1s_{23}s_5]$$
$$-r_{31}(s_{23}c_4c_5 + c_{23}s_5).$$

Differential Motions and Velocities

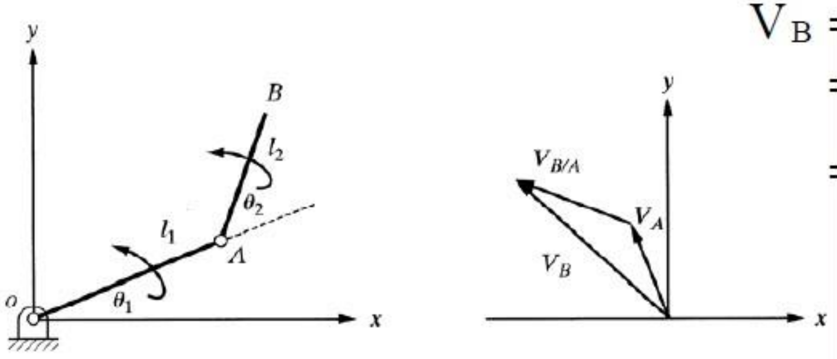
Chapter 3, Introduction to Robotics, Saeed B. Niku

Differential Motions and Velocities

- » Differential motions are small movements of mechanisms (e.g., robots) that can be used to derive velocity relationships between different parts of the mechanism.
- » A differential motion is, by definition, a small movement. Therefore, if it is measured in—or calculated for—a small period of time (a differential or small time), a velocity relationship can be found.

Differential Relationship

» The rotation of the first link θ_1 is measured relative to the reference frame, whereas the rotation of the second link θ_2 is measured relative to the first link. This would be similar to a robot, where each link's movement is measured relative to a current frame attached to the previous link.



(a) (b)

$$
\begin{aligned}
\overline{V}_B &= \overline{V}_A + \overline{V}_{B/A} \\
&= \ell_1\dot{\theta}_1[\perp \text{ to } \ell_1] + \ell_2(\dot{\theta}_1+\dot{\theta}_2)[\perp \text{ to } \ell_2] \\
&= -\ell_1\dot{\theta}_1 \sin\theta_1 \hat{i} + \ell_1\dot{\theta}_1\cos\theta_1 \hat{j} - \ell_2(\dot{\theta}_1+\dot{\theta}_2) \\
&\quad \times \sin(\theta_1+\theta_2)\hat{i} + \ell_2(\dot{\theta}_1+\dot{\theta}_2)\cos(\theta_1+\theta_2)\hat{j}
\end{aligned}
$$

→ matrix form

$$
\begin{bmatrix} \overline{V}_{B_x} \\ \overline{V}_{B_y} \end{bmatrix} = \begin{bmatrix} -\ell_1\sin\theta_1 - \ell_2\sin(\theta_1+\theta_2) & -\ell_2\sin(\theta_1+\theta_2) \\ \ell_1\cos\theta_1 + \ell_2\cos(\theta_1+\theta_2) & \ell_2\cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}
$$

Differential Relationship

→ point B
$$\begin{cases} x_B = \ell_1 \cos\theta_1 + \ell_2 \cos(\theta_1+\theta_2) \\ y_B = \ell_1 \sin\theta_1 - \ell_2 \sin(\theta_1+\theta_2) \end{cases}$$

→ differentiating
$$\begin{cases} dx_B = -\ell_1 \sin\theta_1 d\theta_1 - \ell_2 \sin(\theta_1+\theta_2)(d\theta_1 + d\theta_2) \\ dy_B = \ell_1 \cos\theta_1 d\theta_1 + \ell_2 \cos(\theta_1+\theta_2)(d\theta_1 + d\theta_2) \end{cases}$$

→ matrix form
$$\underbrace{\begin{bmatrix} dx_B \\ dy_B \end{bmatrix}}_{\text{differential motion of B}} = \underbrace{\begin{bmatrix} -\ell_1 \sin\theta_1 - \ell_2 \sin(\theta_1+\theta_2) & -\ell_2 \sin(\theta_1+\theta_2) \\ \ell_1 \cos\theta_1 + \ell_2 \cos(\theta_1+\theta_2) & \ell_2 \cos(\theta_1+\theta_2) \end{bmatrix}}_{\text{Jacobian}} \underbrace{\begin{bmatrix} d\theta_1 \\ d\theta_2 \end{bmatrix}}_{\text{differential motion of joints}}$$

$$\therefore \begin{bmatrix} dx_B \\ dy_B \end{bmatrix} \Big/ dt = \begin{bmatrix} -\ell_1 \sin\theta_1 - \ell_2 \sin(\theta_1+\theta_2) & -\ell_2 \sin(\theta_1+\theta_2) \\ \ell_1 \cos\theta_1 + \ell_2 \cos(\theta_1+\theta_2) & \ell_2 \cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \end{bmatrix} \Big/ dt$$

$$\Leftrightarrow \frac{dx_B}{dt} = \overline{V}_{B_x} \quad \& \quad \frac{dy_B}{dt} = \overline{V}_{B_y}$$

Jacobian

- » Representation of the geometry of the elements of a robot mechanism in time
- » Relationship between the individual joint motions and overall mechanism motions

$Y_i = f_i(x_1, x_2, x_3, \cdots, x_j) \quad \rightarrow \quad$ a set of eq.'s Y_i in terms of a set of variables x_j

Differential change in Y_i for a differential change in x_j

$$
\begin{aligned}
\delta Y_1 &= \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_1}{\partial x_j}\delta x_j \\
\delta Y_2 &= \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_2}{\partial x_j}\delta x_j \\
&\vdots \\
\delta Y_i &= \frac{\partial f_i}{\partial x_1}\delta x_1 + \frac{\partial f_i}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_i}{\partial x_j}\delta x_j
\end{aligned}
$$

Jacobian

→ matrix form

$$\begin{bmatrix} \delta Y_1 \\ \delta Y_2 \\ \vdots \\ \delta Y_i \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_j} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_j} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_i}{\partial x_1} & \frac{\partial f_i}{\partial x_2} & \cdots & \frac{\partial f_i}{\partial x_j} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \delta x_j \end{bmatrix} \quad \text{or} \quad [\delta Y_i] = \left[\frac{\partial f_i}{\partial x_j}\right][\delta x_j]$$

→ robot

$$\begin{bmatrix} \mathrm{dx} \\ \mathrm{dy} \\ \mathrm{dz} \\ \delta\mathrm{x} \\ \delta\mathrm{y} \\ \delta\mathrm{z} \end{bmatrix} = \begin{bmatrix} \text{Robot} \\ \text{Jacobian Matrix} \end{bmatrix} \begin{bmatrix} \mathrm{d}\theta_1 \\ \mathrm{d}\theta_2 \\ \mathrm{d}\theta_3 \\ \mathrm{d}\theta_4 \\ \mathrm{d}\theta_5 \\ \mathrm{d}\theta_6 \end{bmatrix} \quad \text{or} \quad [D] = [J][D_\theta]$$

dx, dy, dz → differential motions of the hand along the axis

δx , δy, δz → differential rotations of the hand w.r.t. the axis

Jacobian

Ex 3.1) Given the joint differential motions $[D_\theta]$ and the Jacobian [J], compute the linear and angular differential motions [D].

$$J = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathrm{D}_\theta = \begin{bmatrix} 0 \\ 0.1 \\ -0.1 \\ 0 \\ 0 \\ 0.2 \end{bmatrix}$$

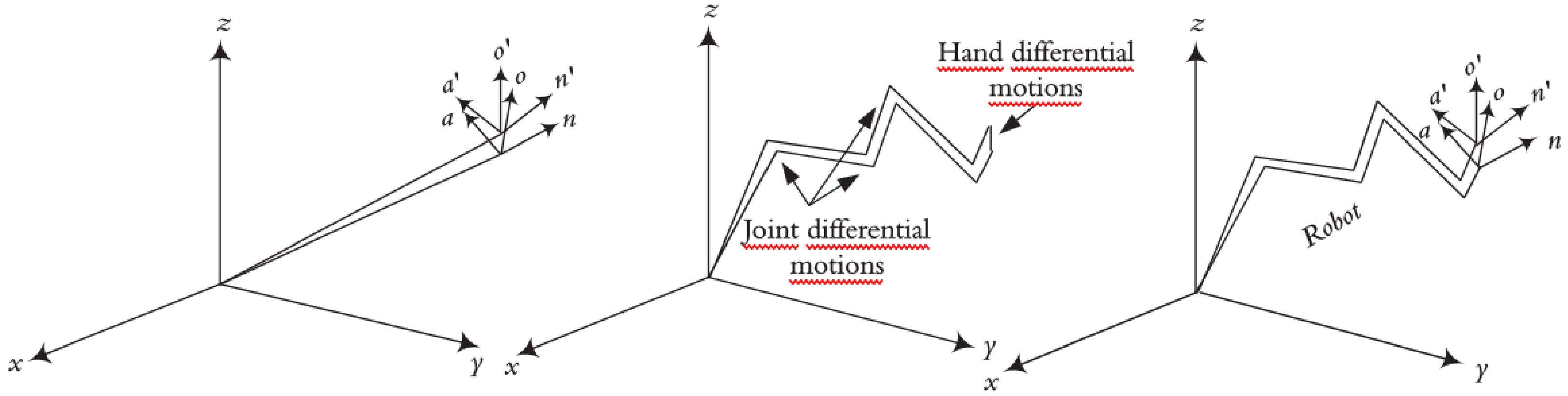
$$\Rightarrow \quad \mathrm{D} = J\mathrm{D}_\theta = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.1 \\ -0.1 \\ 0 \\ 0 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.1 \\ 0.1 \\ 0 \\ -0.1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} \mathrm{dx} \\ \mathrm{dy} \\ \mathrm{dz} \\ \delta\mathrm{x} \\ \delta\mathrm{y} \\ \delta\mathrm{z} \end{bmatrix}$$

Differential Motions of a Frame versus a Robot

Differential motions of a frame,

Differential motions of the robot joints and the end- plate,

Differential motions of a frame caused by the differential motions of a robot.



Differential Motions of a Frame

Differential motions of a frame can be divided into the following

- Differential translations,
- Differential rotations
- Differential transformations (combinations of translations and rotations)

Differential Translations

A differential translation is the translation of a frame at differential values. Therefore, it can be represented by Trans(dx, dy, dz). This means the frame has moved a differential amount along the x-, y-, and z-axes.

Differential Translations

E 3.2: A frame B has translated a differential amount of Trans(0.01, 0.05, 0.03) units. Find its new location and orientation.

$$B = \begin{bmatrix} 0.707 & 0 & -0.707 & 5 \\ 0 & 1 & 0 & 4 \\ 0.707 & 0 & 0.707 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0.01 \\ 0 & 1 & 0 & 0.05 \\ 0 & 0 & 1 & 0.03 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0.707 & 0 & -0.707 & 5 \\ 0 & 1 & 0 & 4 \\ 0.707 & 0 & 0.707 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.707 & 0 & -0.707 & 5.01 \\ 0 & 1 & 0 & 4.05 \\ 0.707 & 0 & 0.707 & 9.03 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Differential rotations about the reference axes

- A differential rotation is a small rotation of the frame.
- It is generally represented by $Rot(q, d\theta)$, means that the frame has rotated an angle of $d\theta$ about an axis q.

Differential rotations about the reference axes

- » Differential Rotations
  (δx, δy, δz)
- » Use approximations → sin δx= δx, cosδx= 1
- » If we do neglect the higher-order differentials such as $(\delta x)^2$, the magnitude of the vectors remain acceptable.

$$\mathrm{Rot}(x,\delta x)=\begin{bmatrix}1&0&0&0\\0&1&-\delta x&0\\0&\delta x&1&0\\0&0&0&1\end{bmatrix}\quad \mathrm{Rot}(y,\delta y)=\begin{bmatrix}1&0&\delta y&0\\0&1&0&0\\-\delta y&0&1&0\\0&0&0&1\end{bmatrix}\quad \mathrm{Rot}(z,\delta z)=\begin{bmatrix}1&-\delta z&0&0\\\delta z&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

Differential rotations about the reference axes

- $\mathrm{Rot}(x,\delta x)\mathrm{Rot}(y,\delta y) = \mathrm{Rot}(y,\delta y)\mathrm{Rot}(x,\delta x) \rightarrow$ Commutative?

$$\mathrm{Rot}(x,\delta x)\mathrm{Rot}(y,\delta y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\delta x & 0 \\ 0 & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \delta y & 0 \\ 0 & 1 & 0 & 0 \\ -\delta y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \delta y & 0 \\ \delta x\delta y & 1 & -\delta x & 0 \\ -\delta y & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Rot}(y,\delta y)\mathrm{Rot}(x,\delta x) = \begin{bmatrix} 1 & 0 & \delta y & 0 \\ 0 & 1 & 0 & 0 \\ -\delta y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\delta x & 0 \\ 0 & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \delta x\delta y & \delta y & 0 \\ 0 & 1 & -\delta x & 0 \\ -\delta y & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \text{Yes, commutative} \Rightarrow \quad \mathrm{Rot}(x,\delta x)\mathrm{Rot}(y,\delta y) = \mathrm{Rot}(y,\delta y)\mathrm{Rot}(x,\delta x) = \begin{bmatrix} 1 & 0 & \delta y & 0 \\ 0 & 1 & -\delta x & 0 \\ -\delta y & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Differential rotations about a general axis k

- Differential Rotation about a General Axis k
  - Rot(k,δθ)= Rot(x,δx)Rot(y,δy)Rot(z,δz)

$$\mathrm{Rot}(k,d\theta)=\mathrm{Rot}(x,\delta x)\mathrm{Rot}(y,\delta y)\mathrm{Rot}(z,\delta z)$$

$$=\begin{bmatrix}1&0&0&0\\0&1&-\delta x&0\\0&\delta x&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&\delta y&0\\0&1&0&0\\-\delta y&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&-\delta z&0&0\\\delta z&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}1&-\delta z&\delta y&0\\\delta z+\delta x\delta y&1-\delta x\delta y\delta z&-\delta x&0\\-\delta y+\delta x\delta z&\delta x+\delta y\delta z&1&0\\0&0&0&1\end{bmatrix}$$

Neglecting all higher order differentials,

$$\rightarrow\ \mathrm{Rot}(k,d\theta)=\mathrm{Rot}(x,\delta x)\mathrm{Rot}(y,\delta y)\mathrm{Rot}(z,\delta z)=\begin{bmatrix}1&-\delta z&\delta y&0\\\delta z&1&-\delta x&0\\-\delta y&\delta x&1&0\\0&0&0&1\end{bmatrix}$$

Differential transformation

E 3.3: Find the total differential transformation caused by small rotations about the three axes of $\delta x = 0.1, \delta y = 0.05, \delta z = 0.02$ radians.

$$Rot(q, d\theta) = \begin{bmatrix} 1 & -\delta z & \delta y & 0 \\ \delta z & 1 & -\delta x & 0 \\ -\delta y & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -0.02 & 0.05 & 0 \\ 0.02 & 1 & -0.1 & 0 \\ -0.05 & 0.1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Differential Transformation of a frame

- Differential Transformation of a Frame
  - Combination of differential translations and rotations
    - T = the original frame, dT = the change in T
    - $[T+dT] = [Trans(dx,dy,dz)Rot(k,d\theta)][T]$
  - D/ $[dT]$ = Differential Transformation
    - $[dT] = [Trans(dx,dy,dz)Rot(k,d\theta) - I][T]$
  - D/ Δ = Differential Operator
    - $[dT] = [\Delta][T]$
    - $[\Delta] = [Trans(dx,dy,dz)Rot(k,d\theta) - I]$

Differential Transformation of a frame

$$\Delta = \text{Trans}(\text{dx}, \text{dy}, \text{dz}) \times \text{Rot}(k, d\theta) - \text{I}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \text{dx} \\ 0 & 1 & 0 & \text{dy} \\ 0 & 0 & 1 & \text{dz} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\delta\text{z} & \delta\text{y} & 0 \\ \delta\text{z} & 1 & -\delta\text{x} & 0 \\ -\delta\text{y} & \delta\text{x} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\delta\text{z} & \delta\text{y} & \text{dx} \\ \delta\text{z} & 0 & -\delta\text{x} & \text{dy} \\ -\delta\text{y} & \delta\text{x} & 0 & \text{dz} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

→ Differential operator Δ is not a transformation matrix, nor a frame

Differential transformation of a frame

E 3.4: Write the differential operator matrix for the following differential transformations: $dx = 0.05, dy = 0.03, dz = 0.01$ units and $\delta x = 0.02, \delta y = 0.04, \delta z = 0.06$ radians.

$$\mathbf{\Delta} = \begin{bmatrix} 0 & -0.06 & 0.04 & 0.05 \\ 0.06 & 0 & -0.02 & 0.03 \\ -0.04 & 0.02 & 0 & 0.01 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thank you!