INDUSTRIAL AUTOMATION & ROBOTICS TECHNOLOGY

Inverse Kinematics

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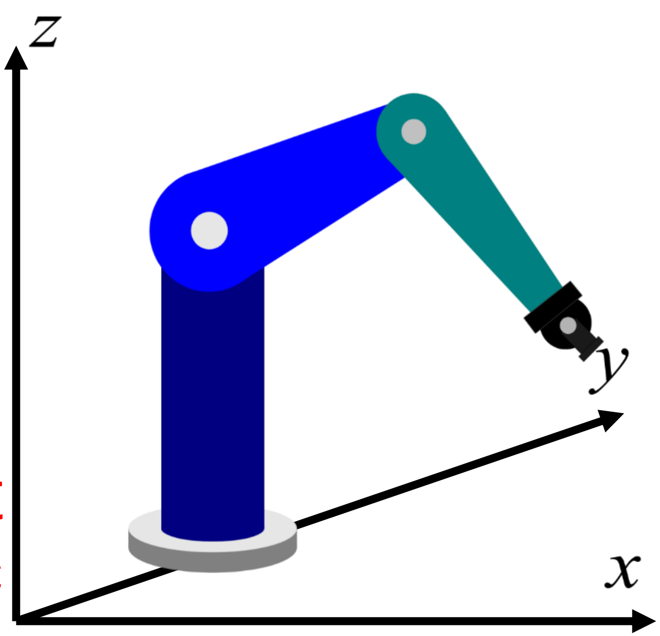
Materials used

- Chapter 4 Introduction to Robotics, John J. Craig
- Chapter 3, Fundamentals of Robotics, Analysis and Control, Robert Schilling

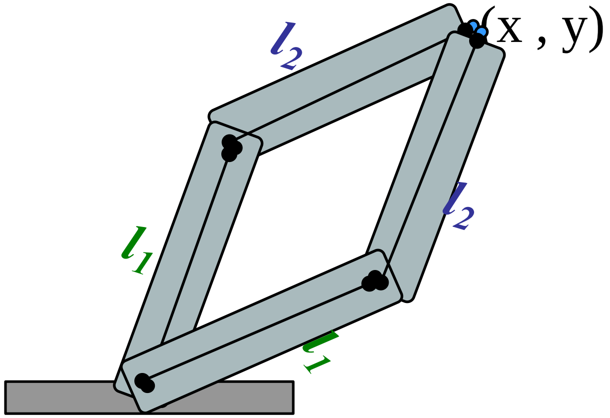
- Given a desired position (P) & orientation (R) of the end-effector

$$q = (q_1, q_2, \ldots, q_n)$$

- Find the joint variables which can bring the robot the desired configuration
- The inverse Kinematics problem is more difficult than the direct Kinematics because a systematic closed-form solution applicable to robots in general is not available.



- More difficult
  - Systematic closed-form solution in general is not available
  - Solution not unique
    - Redundant robot
    - Elbow-up/elbow-down configuration
  - Robot dependent



- Transformation Matrix

$$T = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = T_0^1 T_1^2 T_2^3 T_3^4 T_4^5 T_5^6 \Rightarrow \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{bmatrix}$$

Suppose Q represents the range of values in Rn that the joint variable can assume Q is referred to as the joint-space work envelope of the robot.

Typically the joint-space work envelope has the following general form

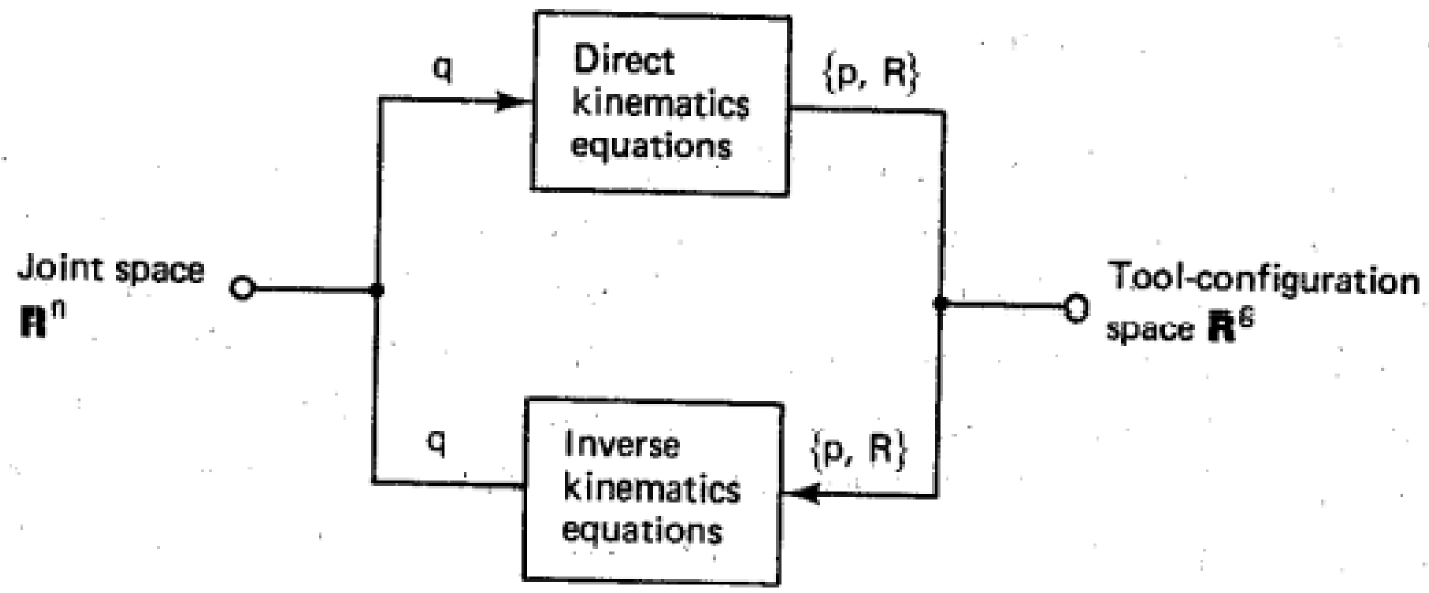
$$Q = \{q \in \mathbb{R}^n : q^{min} \le Cq \le q^{max}\}$$

q^{min} and q^{max} represent joint limits and C is joint coupling matrix.

Inverse Kinematics

- Vector space $\mathbb{R}^n$ is known as joint space
- Tool configuration parameters $\{R, p\}$ can be associated with a subset W of $\mathbb{R}^6$
- Vector space $\mathbb{R}^6$ is referred as tool-configuration space
- Tool-configuration space is six-dimensional because arbitrary configurations of the tool can be specified by using three position coordinates together with three orientation coordinates

FK and IK


Direct kinematics equations
q
{p, R}
Joint space $\mathbb{R}^n$
Tool-configuration space $\mathbb{R}^6$
Inverse kinematics equations
q
{p, R}

General properties of solution

» Existence of solutions

» If the desired tool-tip position p is outside its work envelope, then no solution can exist.

» Even when p is within the work envelope, there may be certain tool orientations R which are not realizable

General properties of solution

- The last row of the arm matrix is always constant
- The arm equation constitutes a system of 12 simultaneous nonlinear algebraic equations in the n unknown components of q.

$$T_{\text{base}}^{\text{tool}}(q) = \left[\begin{array}{ccc|c} R_{11} & R_{12} & R_{13} & p_1 \\ R_{21} & R_{22} & R_{23} & p_2 \\ R_{31} & R_{32} & R_{33} & p_3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

General properties of solution

» Three columns of R form an orthonormal set

» $r^1 \cdot r^2 = 0$

» $r^1 \cdot r^3 = 0$

» $r^2 \cdot r^3 = 0$

» These constraints come from the off-diagonal terms of $R^T R = I$

Inverse Kinematics

- The general strategy for solving the IK problem simplifies somewhat when the robot has a spherical wrist
- Consider the case of an n-axis robot, where $4 \leq n \leq 6$
- Suppose the last axis is a tool roll axis and suppose the robot has a spherical wrist, which means that n-3 axes at the end of the arm all intersect at a point.
- For this class, IK problem can be decomposed into two smaller subproblems.
- Given the tool tip position p and tool orientation R, the wrist position can be inferred from p by working backward along the approach vector

$$p^{wrist} = p - d_n r^3$$

- d_n represents the tool length for an n-axis robot as long as the last axis is a tool roll axis.
- The approach vector r^3 is simply the third column of the rotation matrix R

IK

» Once the wrist position p^{wrist} is obtained from $\{p, R, d_n\}$, the first three joint variables $\{q_1, q_2, q_3\}$ that are used to position the wrist can be obtained from the reduced arm equation

$$T_{\text{base}}^{\text{wrist}}(q_1,\ q_2,\ q_3)i^4 = \begin{bmatrix} p - d_n r^3 \\ 1 \end{bmatrix}$$

Tool Configuration

» p represents the tool position relative to the base

» R represents the tool orientation relative to the base

» Tool Configuration vector

– Tool orientation provided by the approach vector.

– Approach vector specifies both the tool yaw angle and the tool pitch angle, but no the tool roll angle

Tool Configuration

» Tool configuration vector

– To recover the tool roll angle from a scaled approach vector, we must use an invertible function of the roll angle q_n to scale the length of r^3

– The following positive, invertible, exponential scaling function can be used

$$f\left(q_n\right) \triangleq \exp\left(\frac{q_n}{\pi}\right)$$

Tool Configuration

» Tool configuration vector

– is a vector w in R^6

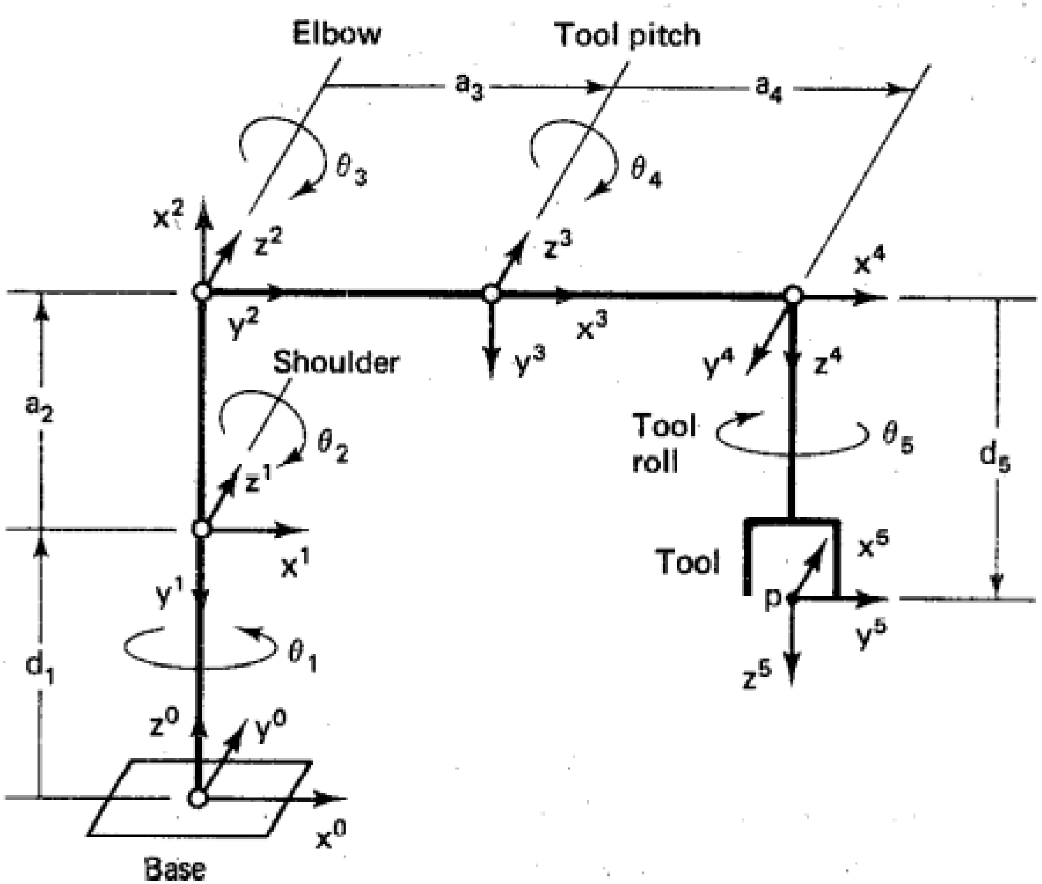
$$w \triangleq \begin{bmatrix} w^1 \\ w^2 \end{bmatrix} \triangleq \begin{bmatrix} p \\ \left[\exp\left(q_n / \pi\right)\right] r^3 \end{bmatrix}$$

- Tool Roll angle

$$q_n = \pi \ln\left(w_4^2 + w_5^2 + w_6^2\right)^{1/2}$$

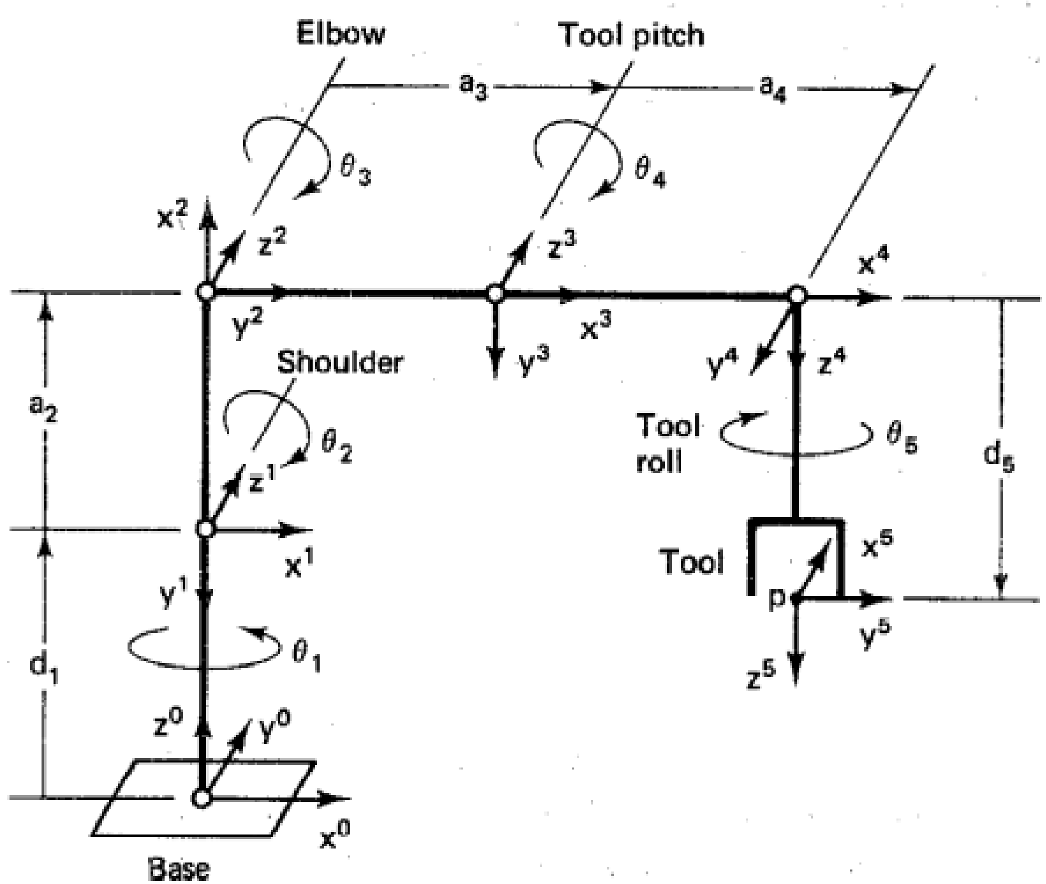
IK of a five-axis articulated robot

» Solution can be approached either numerically or analytically



IK of a five-axis articulated robot

» Solution can be approached either numerically or analytically



IK of a five-axis articulated robot

» Arm matrix

$$\left[\begin{array}{ccc|c} C_1 C_{234} C_5 + S_1 S_5 & -C_1 C_{234} S_5 + S_1 C_5 & -C_1 S_{234} & C_1 (a_2 C_2 + a_3 C_{23} + a_4 C_{234} - d_5 S_{234}) \\ S_1 C_{234} C_5 - C_1 S_5 & -S_1 C_{234} S_5 - C_1 C_5 & -S_1 S_{234} & S_1 (a_2 C_2 + a_3 C_{23} + a_4 C_{234} - d_5 S_{234}) \\ -S_{234} C_5 & S_{234} S_5 & -C_{234} & d_1 - a_2 S_2 - a_3 S_{23} - a_4 S_{234} - d_5 C_{234} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$r^3 = -i^3$$

$$w \triangleq \begin{bmatrix} w^1 \\ w^2 \end{bmatrix} \triangleq \begin{bmatrix} p \\ [\exp(q_n/\pi)] r^3 \end{bmatrix}$$

IK of a five-axis articulated robot

Tool configuration vector

$$\left[\begin{array}{ccc|c} C_1C_{234}C_5 + S_1S_5 & -C_1C_{234}S_5 + S_1C_5 & -C_1S_{234} & C_1(a_2C_2 + a_3C_{23} + a_4C_{234} - d_5S_{234}) \\ S_1C_{234}C_5 - C_1S_5 & -S_1C_{234}S_5 - C_1C_5 & -S_1S_{234} & S_1(a_2C_2 + a_3C_{23} + a_4C_{234} - d_5S_{234}) \\ -S_{234}C_5 & S_{234}S_5 & -C_{234} & d_1 - a_2S_2 - a_3S_{23} - a_4S_{234} - d_5C_{234} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$w \triangleq \begin{bmatrix} w^1 \\ w^2 \end{bmatrix} \triangleq \begin{bmatrix} p \\ \left[\exp\left(q_n/\pi\right)\right] r^3 \end{bmatrix}$$

IK of a five-axis articulated robot

Tool configuration vector

$$\left[\begin{array}{ccc:c} C_1C_{234}C_5 + S_1S_5 & -C_1C_{234}S_5 + S_1C_5 & -C_1S_{234} & C_1(a_2C_2 + a_3C_{23} + a_4C_{234} - d_5S_{234}) \\ S_1C_{234}C_5 - C_1S_5 & -S_1C_{234}S_5 - C_1C_5 & -S_1S_{234} & S_1(a_2C_2 + a_3C_{23} + a_4C_{234} - d_5S_{234}) \\ -S_{234}C_5 & S_{234}S_5 & -C_{234} & d_1 - a_2S_2 - a_3S_{23} - a_4S_{234} - d_5C_{234} \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right]$$

$$w(q) = \left[\begin{array}{c} C_1(a_2C_2 + a_3C_{23} + a_4C_{234} - d_5S_{234}) \\ S_1(a_2C_2 + a_3C_{23} + a_4C_{234} - d_5S_{234}) \\ d_1 - a_2S_2 - a_3S_{23} - a_4S_{234} - d_5C_{234} \\ \hdashline -[\exp(q_5/\pi)]C_1S_{234} \\ -[\exp(q_5/\pi)]S_1S_{234} \\ -[\exp(q_5/\pi)]C_{234} \end{array}\right]$$

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Base joint

$$q_1 = \operatorname{atan2}(w_2, w_1)$$

$$w(q) = \begin{bmatrix} C_1(a_2C_2 + a_3C_{23} + a_4C_{234} - d_5S_{234}) \\ S_1(a_2C_2 + a_3C_{23} + a_4C_{234} - d_5S_{234}) \\ d_1 - a_2S_2 - a_3S_{23} - a_4S_{234} - d_5C_{234} \\ \hline -[\exp(q_5/\pi)]C_1S_{234} \\ -[\exp(q_5/\pi)]S_1S_{234} \\ -[\exp(q_5/\pi)]C_{234} \end{bmatrix}$$

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Elbow joint

q_3 is the most difficult joint variable to extract as it is strongly coupled with the shoulder and tool pitch angles

q_{234} is a global tool pitch angle, $q_{234} = q_2 + q_3 + q_4$

$$-\frac{C_1 w_4 + S_1 w_5}{-w_6} = \frac{S_{234}}{C_{234}}$$

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Elbow joint

Once the shoulder angle q_2 and elbow angle q_3 are known, the tool pitch angle q_4 can be computed from q_{234}

$$b_1 = C_1 w_1 + S_1 w_2 - a_4 C_{234} + d_5 S_{234}$$
$$b_2 = d_1 - a_4 S_{234} - d_5 C_{234} - w_3$$

Substituting components of w

$$b_1 = a_2 c_2 + a_3 C_{23}$$
$$b_2 = a_2 S_2 + a_3 S_{23}$$

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Elbow joint

Elbow angle can be isolated by computing

$$\|b\|^2 = a_2^2 + 2a_2a_3C_3 + a_3^2$$

q_3 gives us two solutions, elbow-up and elbow-down

$$q_3 = \pm cos^{-1}\frac{\|b\|^2 - a_2^2 - a_3^2}{2a_2a_3}$$

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Shoulder joint q_2

$$b_1 = a_2 c_2 + a_3 C_{23}$$
$$b_2 = a_2 S_2 + a_3 S_{23}$$

Using sine and cosine trigonometric identities

$$b_1 = (a_2 + a_3 C_3) C_2 - (a_3 S_3) S_2$$
$$b_2 = (a_2 + a_3 C_3) S_2 + (a_3 S_3) C_2$$

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Shoulder joint q_2

$$C_2 = \frac{(a_2 + a_3 C_3) b_1 + (a_3 S_3) b_2}{\|b\|^2}$$

$$S_2 = \frac{(a_2 + a_3 C_3) b_2 - (a_3 S_3) b_1}{\|b\|^2}$$

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Tool pitch joint q_4

$$q_4 = q_{234} - q_2 - q_3$$

Tool roll joint q_5

$$q_5 = \pi \ln\left(w_4^2 + w_5^2 + w_6^2\right)^{1/2}$$

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$$q_1 = \operatorname{atan2}(w_2, w_1)$$

$$q_{234} = \operatorname{atan2}[-(C_1 w_4 + S_1 w_5), -w_6]$$

$$b_1 = C_1 w_1 + S_1 w_2 - a_4 C_{234} + d_5 S_{234}$$

$$b_2 = d_1 - a_4 S_{234} - d_5 C_{234} - w_3$$

$$q_3 = \pm \arccos \frac{\| b \|^2 - a_2^2 - a_3^2}{2 a_2 a_3}$$

$$q_2 = \operatorname{atan2}[(a_2 + a_3 C_3) b_2 - a_3 S_3 b_1, (a_2 + a_3 C_3) b_1 + a_3 S_3 b_2]$$

$$q_4 = q_{234} - q_2 - q_3$$

$$q_5 = \pi \ln (w_4^2 + w_5^2 + w_6^2)^{1/2}$$

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» Algebraic solution

$$ {}^0_6T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ = {}^0_1T(\theta_1)\,{}^1_2T(\theta_2)\,{}^2_3T(\theta_3)\,{}^3_4T(\theta_4)\,{}^4_5T(\theta_5)\,{}^5_6T(\theta_6) $$

» inverse of first frame

$$ [{}^0_1T(\theta_1)]^{-1}\,{}^0_6T = {}^1_2T(\theta_2)\,{}^2_3T(\theta_3)\,{}^3_4T(\theta_4)\,{}^4_5T(\theta_5)\,{}^5_6T(\theta_6). $$

$$ \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^1_6T, $$

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» Forward kinematics solution

$$
{}^1_6T = {}^1_3T\,{}^3_6T = \begin{bmatrix} {}^1r_{11} & {}^1r_{12} & {}^1r_{13} & {}^1p_x \\ {}^1r_{21} & {}^1r_{22} & {}^1r_{23} & {}^1p_y \\ {}^1r_{31} & {}^1r_{32} & {}^1r_{33} & {}^1p_z \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
\begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^1_6T,
$$

$$
\begin{aligned}
{}^1r_{11} &= c_{23}[c_4c_5c_6 - s_4s_6] - s_{23}s_5s_6, \\
{}^1r_{21} &= -s_4c_5c_6 - c_4s_6, \\
{}^1r_{31} &= -s_{23}[c_4c_5c_6 - s_4s_6] - c_{23}s_5c_6, \\
{}^1r_{12} &= -c_{23}[c_4c_5s_6 + s_4c_6] + s_{23}s_5s_6, \\
{}^1r_{22} &= s_4c_5s_6 - c_4c_6, \\
{}^1r_{32} &= s_{23}[c_4c_5s_6 + s_4c_6] + c_{23}s_5s_6, \\
{}^1r_{13} &= -c_{23}c_4s_5 - s_{23}c_5, \\
{}^1r_{23} &= s_4s_5, \\
{}^1r_{33} &= s_{23}c_4s_5 - c_{23}c_5, \\
{}^1p_x &= a_2c_2 + a_3c_{23} - d_4s_{23}, \\
{}^1p_y &= d_3, \\
{}^1p_x &= -a_3s_{23} - a_2s_2 - d_4c_{23}.
\end{aligned}
$$

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» By comparing (2,4) on both sides

$$-s_1 p_x + c_1 p_y = d_3.$$

» To solve an equation of this form, we make the trigonometric substitutions

$$p_x = \rho \cos\phi,$$
$$p_y = \rho \sin\phi,$$

$$c_1 s_\phi - s_1 c_\phi = \frac{d_3}{\rho}.$$

$$\rho = \sqrt{p_x^2 + p_y^2},$$
$$\phi = \operatorname{Atan2}(p_y, p_x).$$

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» Using difference of angles

$$\sin(\phi - \theta_1) = \frac{d_3}{\rho}.$$

$$\cos(\phi - \theta_1) = \pm\sqrt{1 - \frac{d_3^2}{\rho^2}},$$

$$\phi - \theta_1 = \mathrm{Atan2}\left(\frac{d_3}{\rho}, \pm\sqrt{1 - \frac{d_3^2}{\rho^2}}\right).$$

$$\theta_1 = \mathrm{Atan2}(p_y, p_x) - \mathrm{Atan2}\left(d_3, \pm\sqrt{p_x^2 + p_y^2 - d_3^2}\right).$$

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» Equating (1,4)

$$c_1 p_x + s_1 p_y = a_3 c_{23} - d_4 s_{23} + a_2 c_2,$$

» Equating (3,4)

$$-s_1 p_x + c_1 p_y = d_3.$$

$$-p_x = a_3 s_{23} + d_4 c_{23} + a_2 s_2.$$

$$a_3 c_3 - d_4 s_3 = K,$$

$$K = \frac{p_x^2 + p_y^2 + p_x^2 - a_2^2 - a_3^2 - d_3^2 - d_4^2}{2a_2}.$$

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$$\theta_3 = \text{Atan2}(a_3, d_4) - \text{Atan2}(K, \pm\sqrt{a_3^2 + d_4^2 - K^2}).$$

$$[{}^0_3T(\theta_2)]^{-1}\,{}^0_6T = {}^3_4T(\theta_4)\,{}^4_5T(\theta_5)\,{}^5_6T(\theta_6),$$

$$\begin{bmatrix} c_1c_{23} & s_1c_{23} & -s_{23} & -a_2c_3 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 \\ -s_1 & c_1 & 0 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^3_6T,$$

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$$\theta_3 = \mathrm{Atan2}(a_3, d_4) - \mathrm{Atan2}(K, \pm\sqrt{a_3^2 + d_4^2 - K^2}).$$

$$[{}^0_3T(\theta_2)]^{-1}\,{}^0_6T = {}^3_4T(\theta_4)\,{}^4_5T(\theta_5)\,{}^5_6T(\theta_6),$$

$$\begin{bmatrix} c_1c_{23} & s_1c_{23} & -s_{23} & -a_2c_3 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 \\ -s_1 & c_1 & 0 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^3_6T,$$

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» Equating (1,4) and (2,4) from both sides

$$c_1 c_{23} p_x + s_1 c_{23} p_y - s_{23} p_z - a_2 c_3 = a_3,$$
$$-c_1 s_{23} p_x - s_1 s_{23} p_y - c_{23} p_z + a_2 s_3 = d_4.$$

» Solve both equations simultaneously

$$s_{23} = \frac{(-a_3 - a_2 c_3) p_z + (c_1 p_x + s_1 p_y)(a_2 s_3 - d_4)}{p_z^2 + (c_1 p_x + s_1 p_y)^2},$$
$$c_{23} = \frac{(a_2 s_3 - d_4) p_z - (a_3 + a_2 c_3)(c_1 p_x + s_1 p_y)}{p_z^2 + (c_1 p_x + s_1 p_y)^2}.$$

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» Denominators are positive and equal

$$\theta_{23} = \text{Atan2}[(-a_3 - a_2 c_3) p_z - (c_1 p_x + s_1 p_y)(d_4 - a_2 s_3),$$
$$(a_2 s_3 - d_4) p_z - (a_3 + a_2 c_3)(c_1 p_x + s_1 p_y)].$$

$$\theta_2 = \theta_{23} - \theta_3,$$

» Equating (1,3) and (3,3)

$$r_{13} c_1 c_{23} + r_{23} s_1 c_{23} - r_{33} s_{23} = -c_4 s_5,$$
$$-r_{13} s_1 + r_{23} c_1 = s_4 s_5.$$

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» Denominators are positive and equal

$$\theta_{23} = \text{Atan2}[(-a_3 - a_2c_3)p_z - (c_1p_x + s_1p_y)(d_4 - a_2s_3),$$
$$(a_2s_3 - d_4)p_z - (a_3 + a_2c_3)(c_1p_x + s_1p_y)].$$

$$\theta_2 = \theta_{23} - \theta_3,$$

» Equating (1,3) and (3,3)

$$r_{13}c_1c_{23} + r_{23}s_1c_{23} - r_{33}s_{23} = -c_4s_5,$$
$$-r_{13}s_1 + r_{23}c_1 = s_4s_5.$$

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» $s_5 \neq 0$

$$\theta_4 = \text{Atan2}(-r_{13}s_1 + r_{23}c_1, -r_{13}c_1c_{23} - r_{23}s_1c_{23} + r_{33}s_{23}).$$

$$[{}^0_4T(\theta_4)]^{-1}\,{}^0_6T = {}^4_5T(\theta_5){}^5_6T(\theta_6),$$

» Inverse of 0-4 matrix

$$\begin{bmatrix} c_1c_{23}c_4 + s_1s_4 & s_1c_{23}c_4 - c_1s_4 & -s_{23}c_4 & -a_2c_3c_4 + d_3s_4 - a_3c_4 \\ -c_1c_{23}s_4 + s_1c_4 & -s_1c_{23}s_4 - c_1c_4 & s_{23}s_4 & a_2c_3s_4 + d_3c_4 + a_3s_4 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

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» Equating (1,3) and (3,3)

$$r_{13}(c_1c_{23}c_4 + s_1s_4) + r_{23}(s_1c_{23}c_4 - c_1s_4) - r_{33}(s_{23}c_4) = -s_5,$$
$$r_{13}(-c_1s_{23}) + r_{23}(-s_1s_{23}) + r_{33}(-c_{23}) = c_5.$$

$$\theta_5 = \text{Atan2}(s_5, c_5),$$

$$({}^0_5T)^{-1}\,{}^0_6T = {}^5_6T(\theta_6).$$

» Equating (3,1) and (1,1)

$$\theta_6 = \text{Atan2}(s_6, c_6),$$

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$$s_6 = -r_{11}(c_1 c_{23} s_4 - s_1 c_4) - r_{21}(s_1 c_{23} s_4 + c_1 c_4) + r_{31}(s_{23} s_4),$$

$$c_6 = r_{11}[(c_1 c_{23} c_4 + s_1 s_4) c_5 - c_1 s_{23} s_5] + r_{21}[(s_1 c_{23} c_4 - c_1 s_4) c_5 - s_1 s_{23} s_5]$$
$$-r_{31}(s_{23} c_4 c_5 + c_{23} s_5).$$

Thank you!