ME210 STRENGTH OF MATERIALS

CHAPTER 9

Deflection of Beams

Deformation Under Transverse Loading

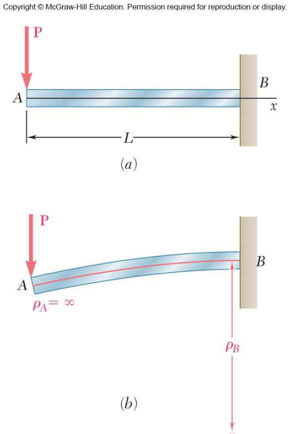


Fig. 9.3 (a) Cantilever beam with concentrated load. (b) Deformed beam showing curvature at ends.

Relationship between bending moment and curvature for pure bending remains valid for general transverse loadings.

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

Cantilever beam subjected to concentrated load **P** at the free end,

$$\frac{1}{\rho} = -\frac{Px}{EI}$$

Curvature varies linearly with x

At the free end A, $\frac{1}{\rho_A} = 0, \quad \rho_A = \infty$

At the support B, $\frac{1}{\rho_B} \neq 0, \quad |\rho_B| = \frac{EI}{PL}$

Deformation Under Transverse Loading

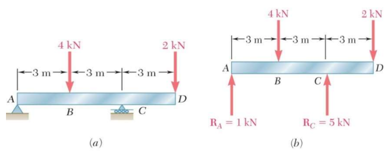


Fig. 9.4 (a) Overhanging beam with two concentrated loads. (b) Free-body diagram showing reaction forces.

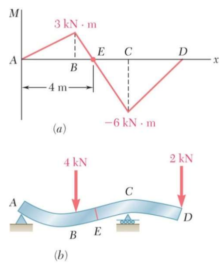


Fig. 9.5 Beam of Fig. 9.4. (a) Bending-moment diagram. (b) Deformed shape.

- Overhanging beam
- Reactions at A and C
- Bending moment diagram
- Curvature is zero at points where the bending moment is zero, i.e., at each end and at E.

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

- Beam is concave upwards where the bending moment is positive and concave downwards where it is negative.
- Maximum curvature occurs where the moment magnitude is a maximum.
- An equation for the beam shape or *elastic curve* is required to determine maximum deflection and slope.

Equation of the Elastic Curve

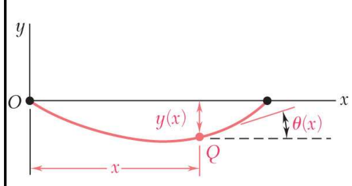


Fig. 9.7 Slope $\theta(x)$ of tangent to the elastic curve.

From elementary calculus, simplified for beam parameters,

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} \approx \frac{d^2y}{dx^2}$$

Substituting and integrating,

$$EI\frac{1}{\rho} = EI\frac{d^2y}{dx^2} = M(x)$$

$$EI\,\theta \approx EI\frac{dy}{dx} = \int_0^x M(x)dx + C_1$$

$$EI\,y = \int_0^x dx \int_0^x M(x)dx + C_1x + C_2$$

Equation of the Elastic Curve

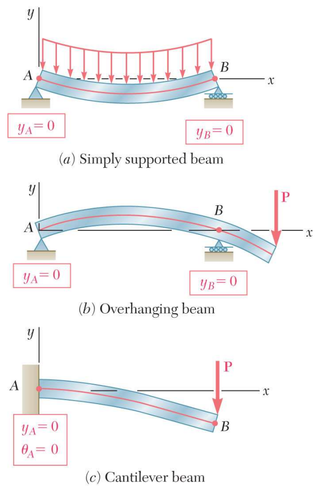


Fig. 9.8 Known boundary conditions for statically determinate beams.

Constants are determined from boundary conditions

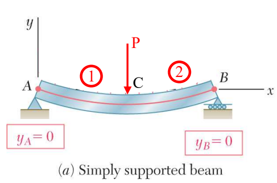
$$EI\,y = \int_0^x dx \int_0^x M(x)\,dx + C_1 x + C_2$$

Three cases for statically determinant beams,

- Simply supported beam
  $y_A = 0, \quad y_B = 0$
- Overhanging beam
  $y_A = 0, \quad y_B = 0$
- Cantilever beam
  $y_A = 0, \quad \theta_A = 0$

More complicated loadings require multiple integrals and application of requirement for continuity of displacement and slope.

Equation of the Elastic Curve



Consider the simply supported beam, on left.

For this case $M(x)$ is defined piecewise. For the part AC and CB we have different expressions and therefore different differential equations. For example if |AC|=|CB|=L/2

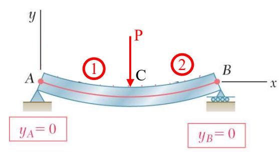
For AC: $M(x) = \frac{Px}{2}$ For CB: $M(x) = \frac{PL}{2} - \frac{Px}{2}$

As a result of integration of $M(x)$ over AC and CB we will get 4 integration constants.

$$EIy = \int_0^{x_C} dx \int_0^{x_C} M(x)dx + C_1 x + C_2 \qquad EIy = \int_{x_C}^{x_B} dx \int_{x_C}^{x_B} M(x)dx + C_3 x + C_4$$

Thus, we need 4 support (boundary) conditions to determine the constants!

Equation of the Elastic Curve



(a) Simply supported beam

2 conditions: $y_A = 0, \quad y_B = 0$

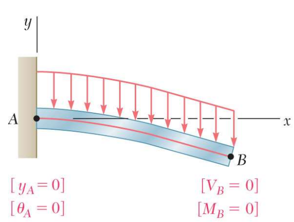
Other 2 conditions are obtained from the compatibility condition at C.

$$y_1(x)\Big|_{x=x_C} = y_2(x)\Big|_{x=x_C}$$

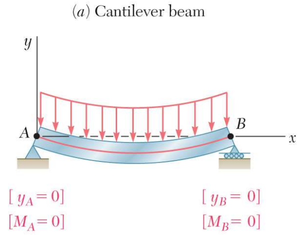
$$\theta_1(x)\Big|_{x=x_C} = \theta_2(x)\Big|_{x=x_C} \qquad \left(\frac{dy_1}{dx} = \frac{dy_2}{dx}\right)_{x=x_C}$$

Deflection and slope cannot be discontinuous.

Determination of Elastic Curve from the Load Distribution



(a) Cantilever beam



(b) Simply supported beam

For a beam subjected to a distributed load,

$$\frac{dM}{dx} = V(x) \qquad \frac{d^2M}{dx^2} = \frac{dV}{dx} = -w(x)$$

Equation for beam displacement becomes

$$\frac{d^2M}{dx^2} = EI\frac{d^4y}{dx^4} = -w(x)$$

Integrating four times yields

$$EI\,y(x) = -\int dx \int dx \int dx \int w(x)dx$$

$$+\tfrac{1}{6}C_1x^3 + \tfrac{1}{2}C_2x^2 + C_3x + C_4$$

Constants are determined from boundary conditions.

Fig. 9.12 Boundary conditions for (a) cantilever beam (b) simply supported beam.

Statically Indeterminate Beams

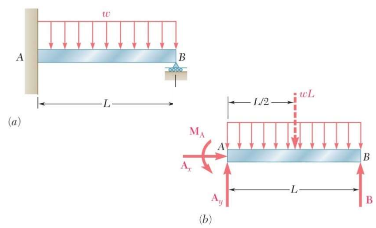


Fig. 9.14 (a) Statically indeterminate beam with a uniformly distributed load.
(b) Free-body diagram with four unknown reactions.

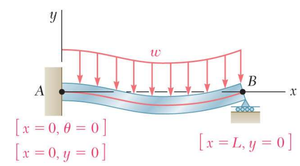


Fig. 9.15 Boundary conditions for beam of Fig. 9.14.

Consider beam with fixed support at *A* and roller support at *B*.

From free-body diagram, note that there are four unknown reaction components.

Conditions for static equilibrium yield

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_A = 0$$

The beam is statically indeterminate.

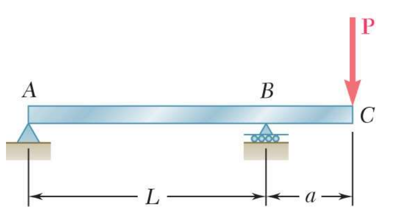
Also have the beam deflection equation,

$$EI\,y = \int_0^x dx \int_0^x M(x)\,dx + C_1 x + C_2$$

which introduces two unknowns but provides three additional equations from the boundary conditions:

At $x = 0,\ \theta = 0\ y = 0$ At $x = L,\ y = 0$

Sample Problem 9.1



$W14\times 68$ $I = 722\,\text{in}^4$ $E = 29\times 10^6\,\text{psi}$

$P = 50\,\text{kips}$ $L = 15\,\text{ft}$ $a = 4\,\text{ft}$

For portion *AB* of the overhanging beam,
(*a*) derive the equation for the elastic curve,
(*b*) determine the maximum deflection,
(*c*) evaluate y_{max}.

SOLUTION:

Develop an expression for M(x) and derive differential equation for elastic curve.

Integrate differential equation twice and apply boundary conditions to obtain elastic curve.

Locate point of zero slope or point of maximum deflection.

Evaluate corresponding maximum deflection.

Sample Problem 9.1

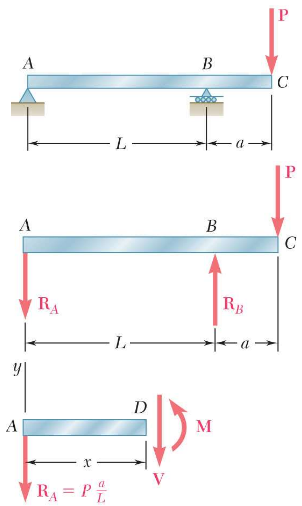


Fig. 1 Free-body diagrams of beam and portion *AD*.

SOLUTION:

Develop an expression for M(x) and derive differential equation for elastic curve.

- Reactions:

$$R_A = \frac{Pa}{L}\downarrow \quad R_B = P\left(1+\frac{a}{L}\right)\uparrow$$

- From the free-body diagram for section *AD*,

$$M = -P\frac{a}{L}x \quad (0 < x < L)$$

- The differential equation for the elastic curve,

$$EI\frac{d^2y}{dx^2} = -P\frac{a}{L}x$$

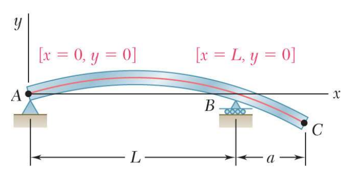


Fig. 2 Boundary conditions.

Integrate differential equation twice and apply boundary conditions to obtain elastic curve.

$$EI\frac{dy}{dx} = -\frac{1}{2}P\frac{a}{L}x^2 + C_1$$

$$EI\,y = -\frac{1}{6}P\frac{a}{L}x^3 + C_1x + C_2$$

at $x = 0,\ y = 0:\quad C_2 = 0$

at $x = L,\ y = 0:\quad EI(0) = -\frac{1}{6}P\frac{a}{L}L^3 + C_1L \quad C_1 = \frac{1}{6}PaL$

Substituting, $$EI\frac{d^2y}{dx^2} = -P\frac{a}{L}x$$

$$EI\frac{dy}{dx} = -\frac{1}{2}P\frac{a}{L}x^2 + \frac{1}{6}PaL \qquad \frac{dy}{dx} = \frac{PaL}{6EI}\left[1 - 3\left(\frac{x}{L}\right)^2\right]$$

$$EI\,y = -\frac{1}{6}P\frac{a}{L}x^3 + \frac{1}{6}PaLx$$

$$y = \frac{PaL^2}{6EI}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^3\right]$$

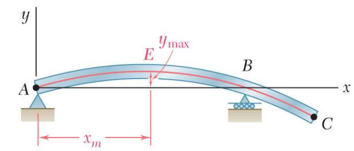


Fig. 3 Deformed elastic curve with location of maximum deflection.

Locate point of zero slope or point of maximum deflection.

$$\frac{dy}{dx} = 0 = \frac{PaL}{6EI}\left[1 - 3\left(\frac{x_m}{L}\right)^2\right] \qquad x_m = \frac{L}{\sqrt{3}} = 0.577L$$

Evaluate corresponding maximum deflection.

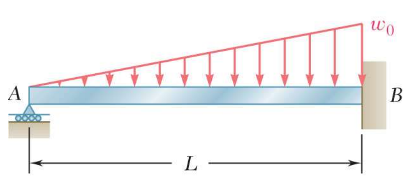
$$y = \frac{PaL^2}{6EI}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^3\right]$$

$$y_{max} = \frac{PaL^2}{6EI}\left[0.577 - (0.577)^3\right] \qquad \boxed{y_{max} = 0.0642\frac{PaL^2}{6EI}}$$

$$y_{max} = 0.0642\frac{(50\,\text{kips})(48\,\text{in})(180\,\text{in})^2}{6\left(29 \times 10^6\,\text{psi}\right)\left(723\,\text{in}^4\right)}$$

$$\boxed{y_{max} = 0.238\,\text{in}}$$

Sample Problem 9.3



For the uniform beam, determine the reaction at A, derive the equation for the elastic curve, and determine the slope at A. (Note that the beam is statically indeterminate to the first degree)

SOLUTION:

Develop the differential equation for the elastic curve (will be functionally dependent on the reaction at A).

Integrate twice and apply boundary conditions to solve for reaction at A and to obtain the elastic curve.

Evaluate the slope at A.

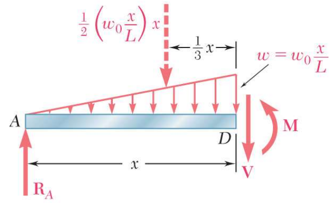


Fig. 1 Free-body diagram of portion *AD* of beam.

Consider moment acting at section D,

$$\sum M_D = 0$$

$$R_A x - \frac{1}{2}\left(\frac{w_0 x^2}{L}\right)\frac{x}{3} - M = 0$$

$$M = R_A x - \frac{w_0 x^3}{6L}$$

The differential equation for the elastic curve,

$$EI\frac{d^2y}{dx^2} = M = R_A x - \frac{w_0 x^3}{6L}$$

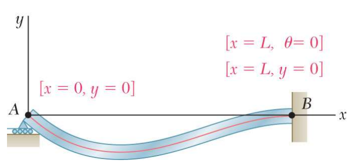


Fig. 2 Boundary conditions.

$$EI\frac{d^2y}{dx^2} = M = R_A x - \frac{w_0 x^3}{6L}$$

Integrate twice

$$EI\frac{dy}{dx} = EI\theta = \frac{1}{2}R_A x^2 - \frac{w_0 x^4}{24L} + C_1$$

$$EI\,y = \frac{1}{6}R_A x^3 - \frac{w_0 x^5}{120L} + C_1 x + C_2$$

Apply boundary conditions:

at $x = 0,\ y = 0:\quad C_2 = 0$

at $x = L,\ \theta = 0:\quad \frac{1}{2}R_A L^2 - \frac{w_0 L^3}{24} + C_1 = 0$

at $x = L,\ y = 0:\quad \frac{1}{6}R_A L^3 - \frac{w_0 L^4}{120} + C_1 L + C_2 = 0$

Solve for reaction at A

$$\frac{1}{3}R_A L^3 - \frac{1}{30}w_0 L^4 = 0 \qquad\qquad R_A = \frac{1}{10}w_0 L \uparrow$$

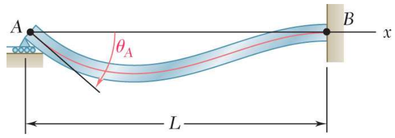


Fig. 3 Deformed elastic curve showing slope at A.

Substitute for C_1, C_2, and R_A in the elastic curve equation,

$$EI\,y = \frac{1}{6}\left(\frac{1}{10}w_0 L\right)x^3 - \frac{w_0 x^5}{120L} - \left(\frac{1}{120}w_0 L^3\right)x$$

$$y = \frac{w_0}{120EIL}\left(-x^5 + 2L^2x^3 - L^4x\right)$$

Differentiate once to find the slope,

$$\theta = \frac{dy}{dx} = \frac{w_0}{120EIL}\left(-5x^4 + 6L^2x^2 - L^4\right)$$

at $x = 0$, $$\theta_A = \frac{w_0 L^3}{120EI}$$

Method of Superposition

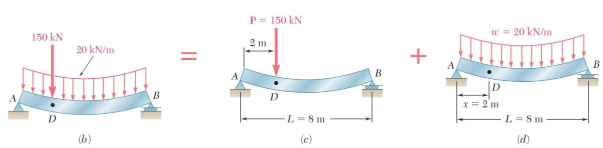


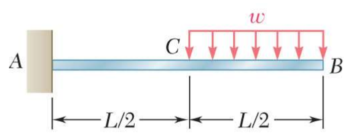
Fig. 9.21b-d (b) The beam's loading can be obtained by superposing deflections due to (c) the concentrated load and (d) the distributed load.

Principle of Superposition:

Deformations of beams subjected to combinations of loadings may be obtained as the linear combination of the deformations from the individual loadings

Procedure is facilitated by tables of solutions for common types of loadings and supports.

Sample Problem 9.7



For the beam and loading shown, determine the slope and deflection at point *B*.

SOLUTION:

Superpose the deformations due to *Loading I* and *Loading II* as shown.

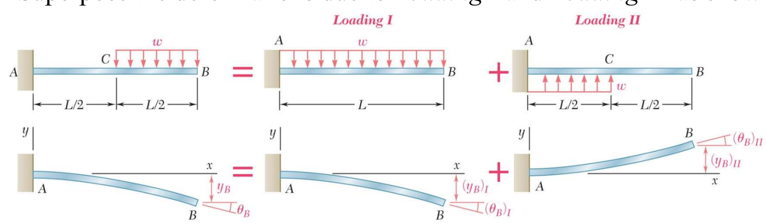


Fig. 1 Actual loading is equivalent to the superposition of two distributed loads.

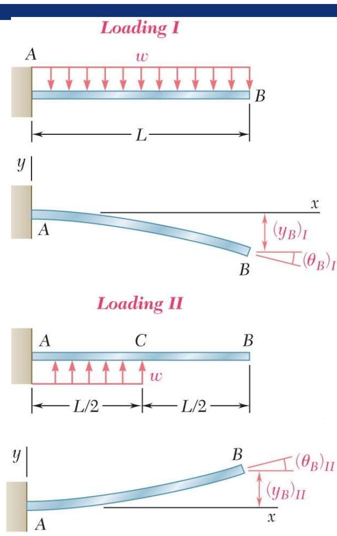


Fig. 2 Deformation details of the superposed loadings I and II.

Loading I

$$(\theta_B)_I = -\frac{wL^3}{6EI} \qquad (y_B)_I = -\frac{wL^4}{8EI}$$

Loading II

$$(\theta_C)_{II} = \frac{wL^3}{48EI} \qquad (y_C)_{II} = \frac{wL^4}{128EI}$$

In beam segment CB, the bending moment is zero and the elastic curve is a straight line.

$$(\theta_B)_{II} = (\theta_C)_{II} = \frac{wL^3}{48EI}$$

$$(y_B)_{II} = \frac{wL^4}{128EI} + \frac{wL^3}{48EI}\left(\frac{L}{2}\right) = \frac{7wL^4}{384EI}$$

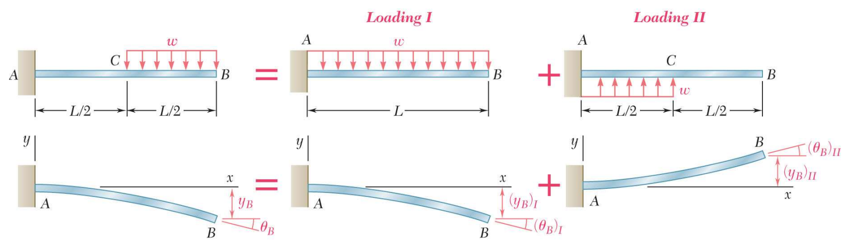


Fig. 1 Actual loading is equivalent to the superposition of two distributed loads.

Combine the two solutions,

$$\theta_B = (\theta_B)_I + (\theta_B)_{II} = -\frac{wL^3}{6EI} + \frac{wL^3}{48EI} \qquad \boxed{\theta_B = -\frac{7wL^3}{48EI}}$$

$$y_B = (y_B)_I + (y_B)_{II} = -\frac{wL^4}{8EI} + \frac{7wL^4}{384EI} \qquad \boxed{y_B = -\frac{41wL^4}{384EI}}$$

Statically Indeterminate Beams

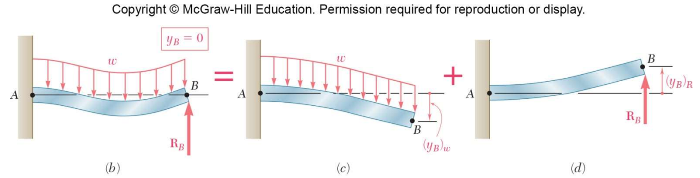


Fig. 9.22 (b) Analyze the indeterminate beam by superposing two determinate cantilever beams, subjected to (c) a uniformly distributed load, (d) the redundant reaction.

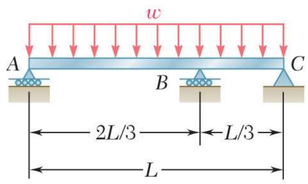
Method of superposition may be applied to determine the reactions at the supports of statically indeterminate beams.

Designate one of the reactions as redundant and eliminate or modify the support.

Determine the beam deformation without the redundant support.

Treat the redundant reaction as an unknown load which, together with the other loads, must produce deformations compatible with the original supports.

Sample Problem 9.8



For the uniform beam and loading shown, determine the reaction at each support and the slope at end *A*.

SOLUTION:

Release the "redundant" support at B, and find deformation.

Apply reaction at *B* as an unknown load to force zero displacement at *B*.

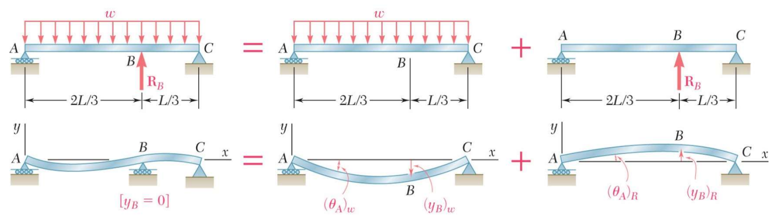
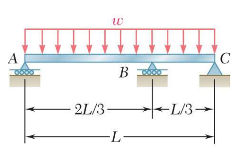
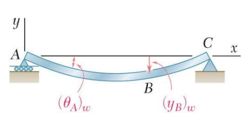


Fig. 1 Indeterminate beam modeled as superposition of two determinate simply supported beams with reaction at *B* chosen redundant.

Sample Problem 9.8





Distributed Loading:

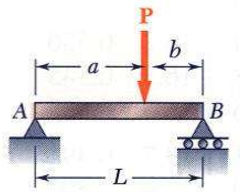
$$(y_B)_w = -\frac{w}{24EI}\left[x^4 - 2Lx^3 + L^3x\right]$$

At point *B*, $x = \frac{2}{3}L$

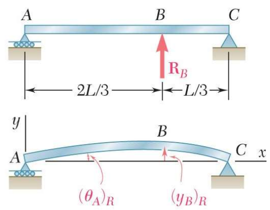
$$(y_B)_w = -\frac{w}{24EI}\left[\left(\frac{2}{3}L\right)^4 - 2L\left(\frac{2}{3}L\right)^3 + L^3\left(\frac{2}{3}L\right)\right]$$

$$= -0.01132\frac{wL^4}{EI}$$

Redundant Reaction Loading:



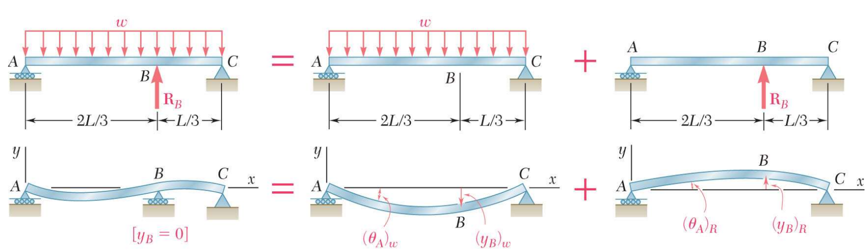
At $x = a, \quad y = -\dfrac{Pa^2b^2}{3EIL}$



For $a = \frac{2}{3}L$ and $b = \frac{1}{3}L$

$$(y_B)_R = \frac{R_B}{3EIL}\left(\frac{2}{3}L\right)^2\left(\frac{L}{3}\right)^2$$

$$= 0.01646\frac{R_B L^3}{EI}$$



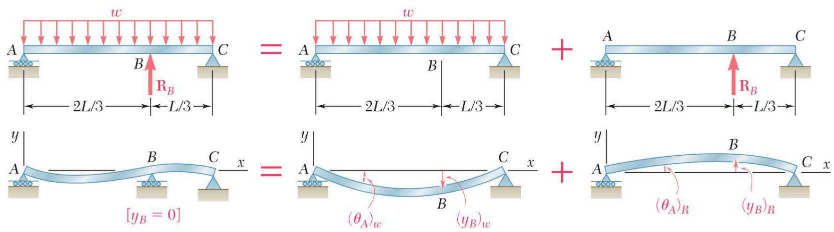
For compatibility with original supports, $y_B = 0$

$$0 = (y_B)_w + (y_B)_R = -0.01132\frac{wL^4}{EI} + 0.01646\frac{R_B L^3}{EI}$$

$$\boxed{R_B = 0.688wL \uparrow}$$

From statics,

$$\boxed{R_A = 0.271wL \uparrow \qquad R_C = 0.0413wL \uparrow}$$



Slope at end A,

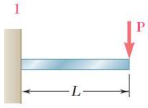
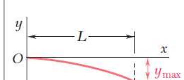
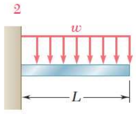
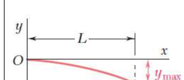
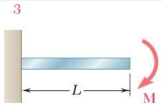
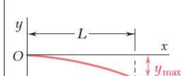
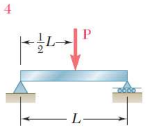
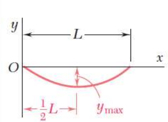
$$(\theta_A)_w = -\frac{wL^3}{24EI} = -0.04167\frac{wL^3}{EI}$$

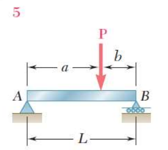
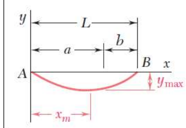
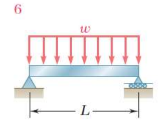
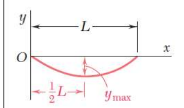
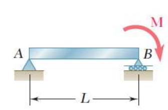
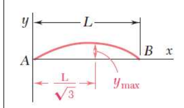
$$(\theta_A)_R = -\frac{Pb\left(L^2 - b^2\right)}{6EIL} = \frac{0.0688wL}{6EIL}\left(\frac{L}{3}\right)\left[L^2 - \left(\frac{L}{3}\right)^2\right] = 0.03398\frac{wL^3}{EI}$$

$$\theta_A = (\theta_A)_w + (\theta_A)_R = -0.04167\frac{wL^3}{EI} + 0.03398\frac{wL^3}{EI}$$

$$\boxed{\theta_A = -0.00769\frac{wL^3}{EI}}$$

APPENDIX D Beam Deflections and Slopes

Beam and Loading	Elastic Curve	Maximum Deflection	Slope at End	Equation of Elastic Curve
1 		$-\frac{PL^3}{3EI}$	$-\frac{PL^2}{2EI}$	$y = \frac{P}{6EI}(x^3 - 3Lx^2)$
2 		$-\frac{wL^4}{8EI}$	$-\frac{wL^3}{6EI}$	$y = -\frac{w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)$
3 		$-\frac{ML^2}{2EI}$	$-\frac{ML}{EI}$	$y = -\frac{M}{2EI}x^2$
4 		$-\frac{PL^3}{48EI}$	$\pm\frac{PL^2}{16EI}$	For $x \le \frac{1}{2}L$: $y = \frac{P}{48EI}(4x^3 - 3L^2x)$

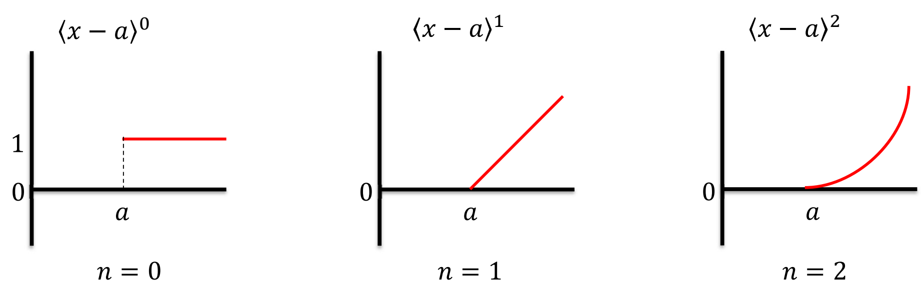
5 		For $a > b$: $-\frac{Pb(L^2-b^2)^{3/2}}{9\sqrt{3}EIL}$ at $x_m = \sqrt{\frac{L^2-b^2}{3}}$	$\theta_A = -\frac{Pb(L^2-b^2)}{6EIL}$ $\theta_B = +\frac{Pa(L^2-a^2)}{6EIL}$	For $x < a$: $y = \frac{Pb}{6EIL}[x^3 - (L^2-b^2)x]$ For $x = a$: $y = -\frac{Pa^2b^2}{3EIL}$
6 		$-\frac{5wL^4}{384EI}$	$\pm \frac{wL^3}{24EI}$	$y = -\frac{w}{24EI}(x^4 - 2Lx^3 + L^3x)$
7 		$\frac{ML^2}{9\sqrt{3}EI}$	$\theta_A = +\frac{ML}{6EI}$ $\theta_B = -\frac{ML}{3EI}$	$y = -\frac{M}{6EIL}(x^3 - L^2x)$

Use of Singularity Functions

Singularity function (for $n \geq 0$)

$$\langle x-a\rangle^n = \begin{cases} (x-a)^n & x \geq a \\ 0 & x < a \end{cases}$$

$$\langle x-a\rangle^0 = \begin{cases} (x-a)^0 = 1 & x \geq a \\ 0 & x < a \end{cases}$$



Use of Singularity Functions

Following relations hold

$$\int \langle x-a\rangle^n\,dx = \frac{1}{n+1}\langle x-a\rangle^{n+1} \quad for \quad n \ge 0$$

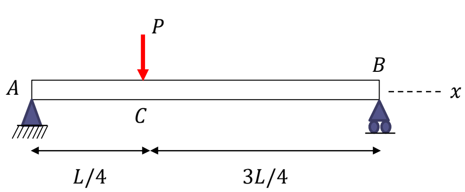
$$\frac{d}{dx}\langle x-a\rangle^n = n\,\langle x-a\rangle^{n-1} \quad for \quad n \ge 1$$

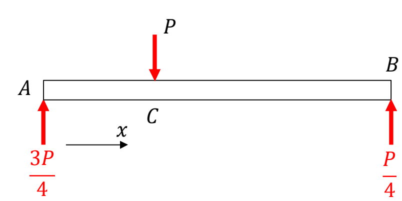
$\langle \ldots \rangle$ whenever the quantity within the bracket is negative, then the bracket is equal to zero.

These functions can help to represent the bending moment $M(x)$ in terms of a single function.

Use of Singularity Functions

Consider the following case:





Between A and C $M_1(x) = \frac{3P}{4}x \qquad \left(0 \le x \le \frac{L}{4}\right)$

Between C and B $M_2(x) = \frac{3P}{4}x - P\left(x - \frac{L}{4}\right) \qquad \left(\frac{L}{4} \le x \le L\right)$

$M(x)$ can also be written as

$$M(x) = \frac{3P}{4}x - P\left\langle x - \frac{L}{4}\right\rangle \qquad (*)$$

$x \ge {}^{L}/_{4}$ then $\left\langle x - \frac{L}{4}\right\rangle = x - \frac{L}{4}$

$x < {}^{L}/_{4}$ then $\left\langle x - \frac{L}{4}\right\rangle = 0$

We can use $(*)$ in 2nd order differential equation

Use of Singularity Functions

$$EI\frac{d^2y}{dx^2} = M(x)$$

$$EI\frac{d^2y}{dx^2} = \frac{3P}{4}x - P\left\langle x - \frac{L}{4}\right\rangle$$

Integrating in x, we have

$$EI\frac{dy}{dx} = \mathrm{EI}\theta(x) = \frac{3}{8}Px^2 - \frac{P}{2}\left\langle x - \frac{L}{4}\right\rangle^2 + c_1 \quad (1)$$

$$EIy = \frac{3}{24}Px^3 - \frac{P}{6}\left\langle x - \frac{L}{4}\right\rangle^3 + c_1x + c_2 \quad (2)$$

c_1 and c_2 can be determined from boundary conditions

$x = 0 \quad y = 0$

$$\boxed{0 = c_2}$$

$x = L \quad y = 0$

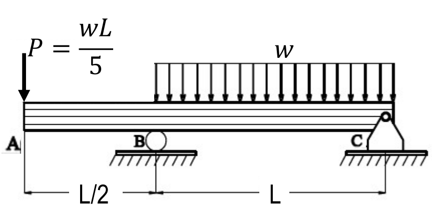
$$0 = \frac{1}{8}PL^3 - \frac{1}{6}P\left\langle\frac{3}{4}L\right\rangle^3 + c_1L$$

$$\left(\frac{16}{128} - \frac{9}{128}\right)PL^3 = -c_1L$$

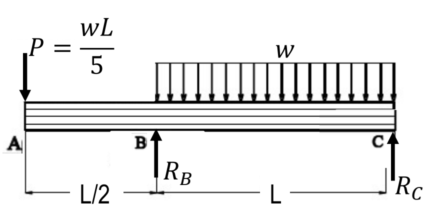
$$\boxed{c_1 = -\frac{7}{128}PL^2}$$

We do not need to determine additional c_3, c_4 which have been determined using slope and displacement continuity at point C. (The continuity of slope and deflection at point C are built in equations (1) and (2)).

Example 1



a) Determine the equation of elastic curve for BC
b) Deflection at the midspan of BC
c) Slope at B



$\sum M_C = 0$ (+↺)

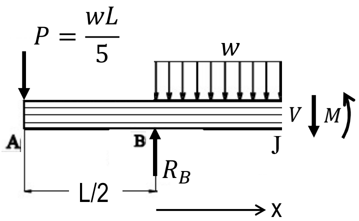
$$\frac{wL}{5}\frac{3L}{2} + wL\frac{L}{2} - R_BL = 0 \qquad \boldsymbol{R_B = \frac{4}{5}wL}$$

$\sum F_y = 0$ (+↓)

$$wL + \frac{wL}{5} - \frac{4wL}{5} - R_C = 0 \qquad \boldsymbol{R_C = \frac{2}{5}wL}$$

Example 1

For BC:



$\sum M_J = 0$ (+)

$$\frac{wL}{5}\left(\frac{L}{2}+x\right)-\frac{4}{5}wLx+(wx)\frac{x}{2}+M=0$$

$$\boldsymbol{M=\frac{3}{5}wLx-\frac{1}{2}wx^2-\frac{1}{10}wL^2}$$

$$EI\frac{d^2y}{dx^2}=\frac{3}{5}wLx-\frac{1}{2}wx^2-\frac{1}{10}wL^2$$

$$EI\frac{dy}{dx}=\frac{3}{10}wLx^2-\frac{1}{6}wx^3-\frac{1}{10}wL^2x+C_1$$

$$EIy=\frac{1}{10}wLx^3-\frac{1}{24}wx^4-\frac{1}{20}wL^2x^2+C_1x+C_2$$

$x=0, y=0 \;\Rightarrow\; \boldsymbol{C_2=0}$

$x=L, y=0 \;\Rightarrow$

$$0=\frac{1}{10}wL^4-\frac{1}{24}wL^4-\frac{1}{20}wL^4+C_1L+0$$

$$\boldsymbol{C_1=-\frac{1}{120}wL^3}$$

Example 1

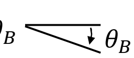
Elastic curve:

a) $$y(x)=\frac{w}{EI}\left(\frac{1}{10}Lx^3-\frac{1}{24}x^4-\frac{1}{20}L^2x^2-\frac{1}{120}L^3x\right)$$

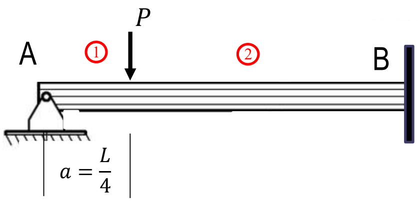
$$\frac{dy}{dx}=\frac{w}{EI}\left(\frac{3}{10}Lx^2-\frac{1}{6}x^3-\frac{1}{10}L^2x-\frac{1}{120}L^3\right)$$

b) $$y\left(x=\frac{L}{2}\right)=\frac{-13wL^4}{1920EI}$$ ↓ y

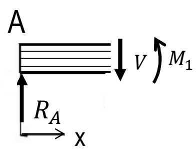
c) $$\frac{dy}{dx}(x=0)=\frac{-wL^3}{120EI}=\theta_B$$



Example 2



Determine the equation of the elastic curve and the deflection at a=L/4.



$$0 \le x \le a$$

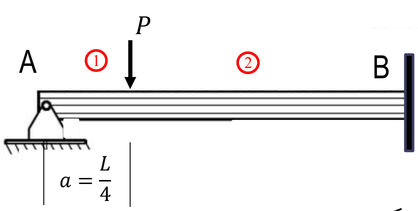
$$M_1 = R_A x$$

$$EI\frac{d^2y}{dx^2} = R_A x \;\Rightarrow\; EI\frac{dy}{dx} = \frac{1}{2}R_A x^2 + C_1$$

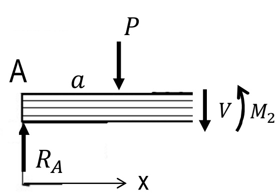
$$EIy = \frac{1}{6}R_A x^3 + C_1 x + C_2$$

$$\boldsymbol{y_1 = \frac{1}{EI}\left(\frac{1}{6}R_A x^3 + C_1 x + C_2\right)} \quad 0 \le x \le \frac{L}{4}$$

Example 2



$$\mathrm{a} \le x \le L$$



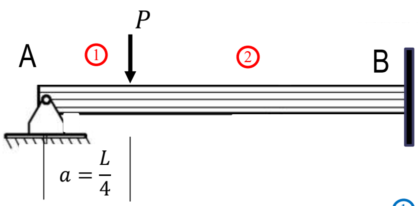
$$M_2 = R_A x - P(x-a)$$

$$EI\frac{d^2y}{dx^2} = M = R_A x - P(x-a)$$

$$\frac{dy}{dx} = \frac{1}{EI}\left(\frac{R_A x^2}{2} - \frac{P(x-a)^2}{2} + C_3\right)$$

$$\boldsymbol{y_2 = \frac{1}{EI}\left(\frac{R_A x^3}{6} - \frac{P(x-a)^3}{6} + C_3 x + C_4\right)}$$

Example 2



BCs

$x = 0,$	$y_1 = 0$ ①
$x = a,$	$y_1' = y_2'$ ②
$x = a,$	$y_1 = y_2$ ③
$x = L,$	$y' = 0$ ④
$x = L,$	$y = 0$ ⑤

$$y_1 = \frac{1}{EI}\left(\frac{1}{6}R_A x^3 + C_1 x + C_2\right)$$

$$y_2 = \frac{1}{EI}\left(\frac{R_A x^3}{6} - \frac{P(x-a)^3}{6} + C_3 x + C_4\right)$$

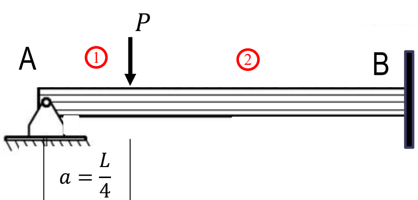
① $\Rightarrow$ $\boldsymbol{C_2 = 0}$

② $\Rightarrow \frac{1}{2}R_A a^2 + C_1 = \frac{1}{2}R_A a^2 + C_3 \quad \boldsymbol{C_1 = C_3}$

③ $\Rightarrow$ $\boldsymbol{C_4 = 0}$

④ $\Rightarrow \boldsymbol{C_3 = \frac{1}{2}P(L-a)^2 - \frac{1}{2}R_A L^2}$

Example 2



BCs

$x = 0,$	$y_1 = 0$ ①
$x = a,$	$y_1' = y_2'$ ②
$x = a,$	$y_1 = y_2$ ③
$x = L,$	$y' = 0$ ④
$x = L,$	$y = 0$ ⑤

$$y_1 = \frac{1}{EI}\left(\frac{1}{6}R_A x^3 + C_1 x + C_2\right)$$

$$y_2 = \frac{1}{EI}\left(\frac{R_A x^3}{6} - \frac{P(x-a)^3}{6} + C_3 x + C_4\right)$$

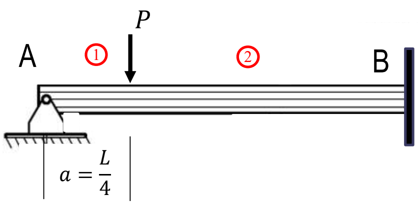
⑤ $\Rightarrow x = L, y_2 = 0$

$$y_2 = \frac{1}{EI}\left(\frac{R_A x^3}{6} - \frac{P(x-a)^3}{6} + \left(\frac{1}{2}P(L-a)^2 - \frac{1}{2}R_A L^2\right)x\right)$$

$$y_2 = \frac{1}{EI}\left(\frac{R_A L^3}{6} - \frac{P(L-a)^3}{6} + \frac{1}{2}P(L-a)^2 x - \frac{1}{2}R_A L^2 x\right) = 0$$

$$R_A\left(\frac{L^3}{6} - \frac{1}{2}L^3\right) = \frac{P}{6}(L-a)^3 - \frac{1}{2}P(L-a)^2 L$$

Example 2



BCs

$x = 0,$	$y_1 = 0$	①
$x = a,$	$y_1' = y_2'$	②
$x = a,$	$y_1 = y_2$	③
$x = L,$	$y' = 0$	④
$x = L,$	$y = 0$	⑤

⑤ $\Rightarrow \; x = L, y_2 = 0$

$$R_A\left(\frac{L^3}{6} - \frac{1}{2}L^3\right) = \frac{P}{6}(L-a)^3 - \frac{1}{2}P(L-a)^2 L$$

$$a = \frac{L}{4}$$

$$R_A\left(\frac{L^3}{6} - \frac{L^3}{2}\right) = \frac{P}{6}\left(L - \frac{L}{4}\right)^3 - \frac{1}{2}P\left(L - \frac{L}{4}\right)^2 L$$

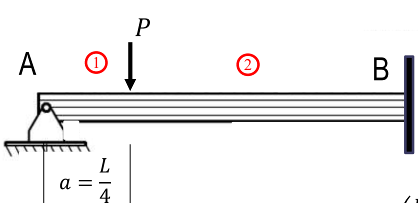
$$R_A\left(-\frac{2L^3}{6}\right) = \frac{P}{6}\left(\frac{3L}{4}\right)^3 - \frac{1}{2}PL\left(\frac{3L}{4}\right)^2$$

$$-R_A\frac{L^3}{3} = \frac{P}{6}27\frac{L^3}{64} - \frac{1}{2}PL\frac{9L^2}{16}$$

$$-\frac{1}{3}R_A L^3 = \frac{9PL^3}{128} - \frac{1}{32}9L^3 \;\Rightarrow\; -\frac{1}{3}R_A L^3 = \left(\frac{9}{128} - \frac{4.9}{128}\right)L^3 P$$

$$R_A = \frac{3}{128}(27) \Rightarrow \boldsymbol{R_A = \frac{81}{128}P}$$

Example 2



BCs

$x = 0,$	$y_1 = 0$	①
$x = a,$	$y_1' = y_2'$	②
$x = a,$	$y_1 = y_2$	③
$x = L,$	$y' = 0$	④
$x = L,$	$y = 0$	⑤

⑤ $\Rightarrow \; x = L, y_2 = 0$

$$R_A\left(\frac{L^3}{6} - \frac{1}{2}L^3\right) = \frac{P}{6}(L-a)^3 - \frac{1}{2}P(L-a)^2 L$$

If $a = L/3$

$$R_A\left(\frac{L^3}{6} - \frac{L^3}{2}\right) = \frac{P}{6}\left(\frac{2L}{3}\right)^3 - \frac{1}{2}P\left(\frac{2L}{3}\right)^2 L$$

$$-R_A\frac{L^3}{3} = \frac{P}{6}\frac{8L^3}{27} - \frac{1}{2}P\frac{4L^3}{9}$$

$$R_A = -3\left(\frac{4}{81} - \frac{2}{9}\right) = -3\left(\frac{4}{81} - \frac{18}{81}\right) \Rightarrow \boldsymbol{R_A = \frac{14}{27}P}$$

If $a = L/3$	If $a = L/4$
$\boldsymbol{R_A = \frac{14}{27}P = 0.5185P}$	$\boldsymbol{R_A = \frac{81}{128}P = 0.6328P}$

Example 2

Equation of elastic curve:

$$y_1 = \frac{1}{EI}\left(\frac{1}{6}0.6328Px^3 + \frac{1}{2}P(L-a)^2x - \frac{1}{2}R_AL^2x\right) \quad a = L/4 \quad R_A = 0.6328P$$

$$y_1 = \frac{1}{EI}\left(0.10547Px^3 + \frac{1}{2}P\left(\frac{3L}{4}\right)^2x - \frac{1}{2}0.6328PL^2x\right)$$

$$\boldsymbol{y_1 = \frac{1}{EI}(0.10547Px^3 + 0.28125PL^2x - 0.3164PL^2x)}$$

$$\boldsymbol{y_2 = \frac{1}{EI}\left(R_A\frac{x^3}{6} - \frac{P}{6}(x-a)^3 + \frac{1}{2}P(L-a)^2x - \frac{1}{2}R_AL^2x\right)}$$

for $x = L/4$

$$y_1 = \frac{1}{EI}\left(0.10547P\left(\frac{L}{4}\right)^3 + 0.28125PL^2\frac{L}{4} - 0.3164PL^2\frac{L}{4}\right)$$

$$y_1 = \frac{1}{EI}(1.648x10^{-3}PL^3 + 0.0703PL^3 - 0.0791PL^3) \quad \boldsymbol{y_1 = -7.152x10^{-3}\frac{P}{EI}L^3}$$