ME210 STRENGTH OF MATERIALS

CHAPTER 3
Torsion

Torsional Loads on Circular Shafts

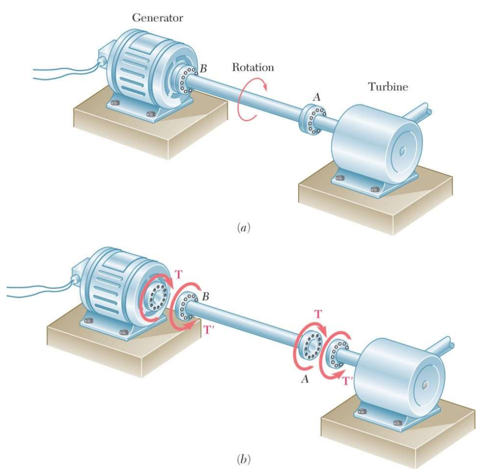


Fig. 3.2 （a） A generator provides power at a constant revolution per minute to a turbine through shaft AB. （b） Free body diagram of shaft AB along with the driving and reaction torques on the generator and turbine, respectively.

Stresses and strains in members of circular cross-section are subjected to twisting couples or *torques*

Turbine exerts torque T on the shaft

Shaft transmits the torque to the generator

Generator creates an equal and opposite torque T'

Net Torque Due to Internal Stresses

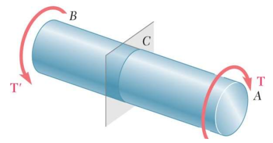


Fig. 3.3 Shaft subject to torques and a section plane at C.

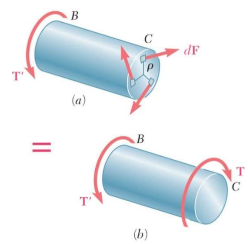


Fig. 3.4 (a) Free body diagram of section BC with torque at C represented by the representable contributions of small elements of area carrying forces dF a radius ρ from the section center. (b) Free-body diagram of section BC having all the small area elements summed resulting in torque T.

Net of the internal shearing stresses is an internal torque, equal and opposite to the applied torque,

$$T = \int \rho \, dF = \int \rho (\tau \, dA)$$

Although the net torque due to the shearing stresses is known, the distribution of the stresses is not.

Distribution of shearing stresses is statically indeterminate – must consider shaft deformations.

Unlike the normal stress due to axial loads, the distribution of shearing stresses due to torsional loads cannot be assumed uniform.

Axial Shear Components

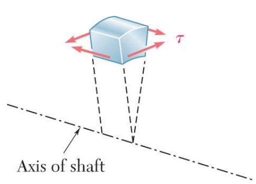


Fig. 3.5 Small element in shaft showing how shear stress components act.

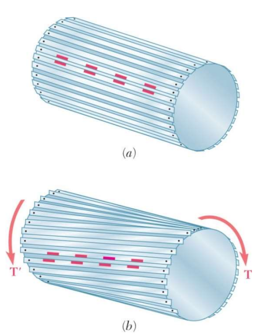


Fig. 3.6 Model of shearing in shaft (a) undeformed; (b) loaded and

Torque applied to shaft produces shearing stresses on the faces perpendicular to the axis.

Conditions of equilibrium require the existence of equal stresses on the faces of the two planes containing the axis of the shaft.

The existence of the axial shear components is demonstrated by considering a shaft made up of slats pinned at both ends to disks.

The slats slide with respect to each other when equal and opposite torques are applied to the ends of the shaft.

Shaft Deformations

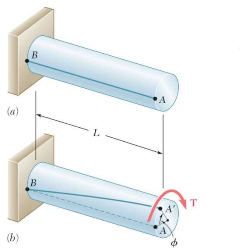


Fig. 3.7 Shaft with fixed support and line AB drawn showing deformation under torsion loading: (a) unloaded; (b) loaded.

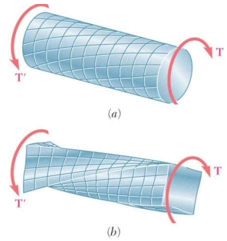


Fig. 3.8 Comparison of deformations in circular (a) and square (b) shafts.

From observation, the *angle of twist* of the shaft is proportional to the applied torque and to the shaft length.

$$\phi \propto T$$

$$\phi \propto L$$

When subjected to torsion, every cross-section of a circular shaft remains plane and undistorted.

Cross-sections for hollow and solid circular shafts remain plain and undistorted because a circular shaft is axisymmetric.

Cross-sections of noncircular (non-axisymmetric) shafts are distorted when subjected to torsion.

Shearing Strain

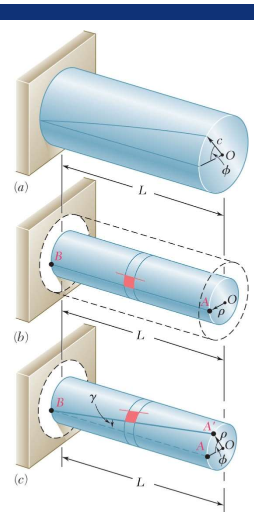


Fig. 3.13 Shearing Strain Kinematic definitions for torsion deformation. (a) The angle of twist ϕ (b) Undeformed portion of shaft of radius ρ with (c) Deformed portion of the shaft having same angle of twist, ϕ and strain, angles of twist per unit length, γ.

Consider an interior section of the shaft. As a torsional load is applied, an element on the interior cylinder deforms into a rhombus.

Since the ends of the element remain planar, the shear strain is equal to angle of twist.

It follows that

$$L\gamma = \rho\phi \quad \text{or} \quad \gamma = \frac{\rho\phi}{L}$$

Shear strain is proportional to twist and radius

$$\gamma_{\max} = \frac{c\phi}{L} \quad \text{and} \quad \gamma = \frac{\rho}{c}\gamma_{\max}$$

Stresses in Elastic Range

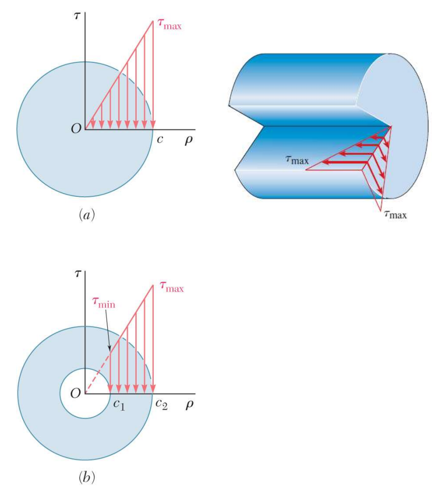


Fig. 3.14 Distribution of shearing stresses in a torqued shaft; (a) Solid shaft, (b) hollow shaft.

Multiplying the previous equation by the shear modulus,

$$G\gamma = \frac{\rho}{c} G\gamma_{max}$$

From Hooke's Law, $\tau = G\gamma$, so

$$\tau = \frac{\rho}{c}\tau_{max}$$

The shearing stress varies linearly with the distance ρ from the axis of the shaft.

Recall that the sum of the moments of the elementary forces exerted on any cross section of the shaft must be equal to the magnitude T of the torque:

$$T = \int \rho\tau\, dA = \frac{\tau_{max}}{c}\int \rho^2\, dA = \frac{\tau_{max}}{c} J$$

The results are known as the *elastic torsion formulas*,

$$\tau_{max} = \frac{Tc}{J} \quad \text{and} \quad \tau = \frac{T\rho}{J}$$

Normal Stresses

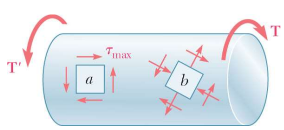


Fig. 3.17 Circular shaft with stress elements at different orientations.

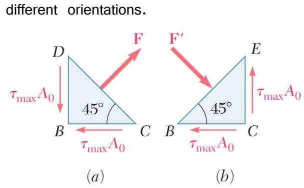


Fig. 3.18 Forces on faces at 45° to shaft axis.

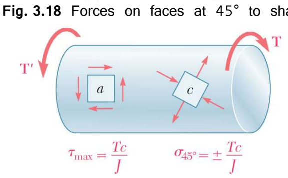


Fig. 3.19 Shaft elements with only shear stresses or normal stresses.

Elements with faces parallel and perpendicular to the shaft axis are subjected to shear stresses only. Normal stresses, shearing stresses or a combination of both may be found for other orientations.

Consider an element at 45° to the shaft axis,

$$F = 2(\tau_{max} A_0)\cos 45^\circ = \tau_{max} A_0 \sqrt{2}$$

$$\sigma_{45^\circ} = \frac{F}{A} = \frac{\tau_{max} A_0 \sqrt{2}}{A_0\sqrt{2}} = \tau_{max}$$

Element *a* is in pure shear.

Element *c* is subjected to a tensile stress on two faces and compressive stress on the other two.

Note that all stresses for elements *a* and *c* have the same magnitude.

Torsional Failure Modes

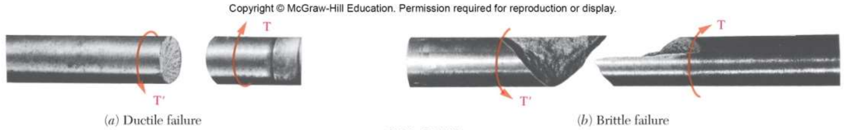


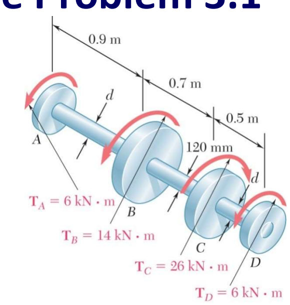
Photo 3.2 Shear failure of shaft subject to torque.

Ductile materials generally fail in shear. Brittle materials are weaker in tension than shear.

When subjected to torsion, a ductile specimen breaks along a plane of maximum shear, i.e., a plane perpendicular to the shaft axis.

When subjected to torsion, a brittle specimen breaks along planes perpendicular to the direction in which tension is a maximum, i.e., along surfaces at 45° to the shaft axis.

Sample Problem 3.1



Shaft *BC* is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts *AB* and *CD* are solid and of diameter *d*. For the loading shown, determine (*a*) the minimum and maximum shearing stress in shaft *BC*, (*b*) the required diameter *d* of shafts *AB* and *CD* if the allowable shearing stress in these shafts is 65 MPa.

SOLUTION:

Cut sections through shafts *AB* and *BC* and perform static equilibrium analyses to find torque loadings.

Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*.

Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter.

SOLUTION:

Cut sections through shafts *AB* and *BC* and perform static equilibrium analysis to find torque loadings.

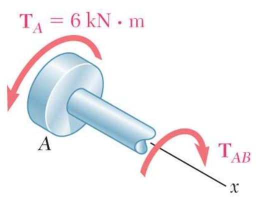


Fig. 1 Free-body diagram for section between A and B.

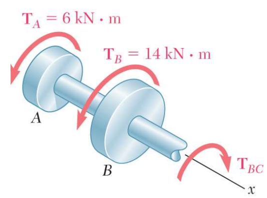


Fig. 2 Free-body diagram for section between B and C.

$$\sum M_x = 0 = (6\,\text{kN}\cdot\text{m}) - T_{AB}$$
$$T_{AB} = 6\,\text{kN}\cdot\text{m} = T_{CD}$$

$$\sum M_x = 0 = (6\,\text{kN}\cdot\text{m}) + (14\,\text{kN}\cdot\text{m}) - T_{BC}$$
$$T_{BC} = 20\,\text{kN}\cdot\text{m}$$

Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*.

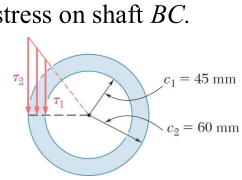


Fig. 3 Shearing stress distribution on cross section.

$$J = \frac{\pi}{2}\left(c_2^4 - c_1^4\right) = \frac{\pi}{2}\left[(0.060)^4 - (0.045)^4\right]$$
$$= 13.92 \times 10^{-6}\,\text{m}^4$$

$$\tau_{max} = \tau_2 = \frac{T_{BC}c_2}{J} = \frac{(20\,\text{kN}\cdot\text{m})(0.060\,\text{m})}{13.92\times10^{-6}\,\text{m}^4}$$
$$= 86.2\,\text{MPa}$$

$$\frac{\tau_{min}}{\tau_{max}} = \frac{c_1}{c_2} \qquad \frac{\tau_{min}}{86.2\,\text{MPa}} = \frac{45\,\text{mm}}{60\,\text{mm}}$$
$$\tau_{min} = 64.7\,\text{MPa}$$

$$\tau_{max} = 86.2\,\text{MPa}$$
$$\tau_{min} = 64.7\,\text{MPa}$$

Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter.

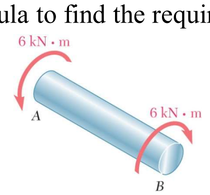


Fig. 4 Free-body diagram of shaft portion AB.

$$\tau_{max} = \frac{Tc}{J} = \frac{Tc}{\frac{\pi}{2}c^4} \qquad 65MPa = \frac{6\,\text{kN}\cdot\text{m}}{\frac{\pi}{2}c^3}$$

$$c = 38.9\times10^{-3}\,\text{m}$$

$$d = 2c = 77.8\,\text{mm}$$

Angle of Twist in Elastic Range

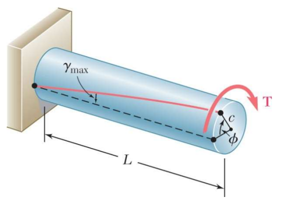


Fig. 3.20 Torque applied to fixed end shaft resulting angle of twist ϕ.

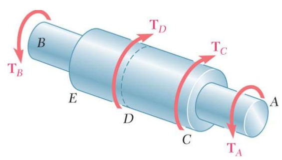


Fig. 3.21 Shaft with multiple cross-section dimensions and multiple loads.

Recall that the angle of twist and maximum shearing strain are related,

$$\gamma_{max} = \frac{c\phi}{L}$$

In the elastic range, the shearing strain and shear are related by Hooke's Law,

$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{Tc}{JG}$$

Equating the expressions for shearing strain and solving for the angle of twist,

$$\phi = \frac{TL}{JG}$$

If the torsional loading or shaft cross-section changes along the length, the angle of rotation is found as the sum of segment rotations

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$

Statically Indeterminate Shafts

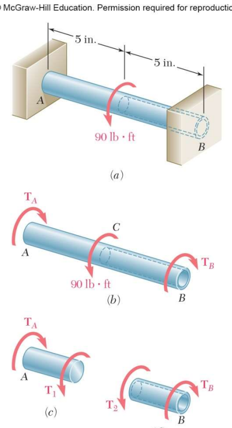


Fig. 3.25 (a) Shaft with central applied torque and fixed ends. (b) free-body diagram of shaft AB. (c) Free-body diagrams for solid and hollow segments.

Given the shaft dimensions and the applied torque, we would like to find the torque reactions at *A* and *B*.

From a free-body analysis of the shaft,

$$T_A + T_B = 90\,\text{lb}\cdot\text{ft}$$

which is not sufficient to find the end torques. The problem is *statically indeterminate*.

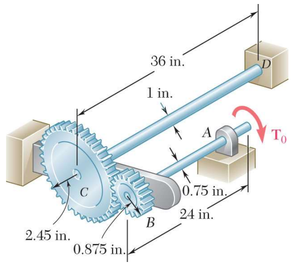
Divide the shaft into two components which must have compatible deformations,

$$\phi = \phi_1 + \phi_2 = \frac{T_A L_1}{J_1 G} - \frac{T_B L_2}{J_2 G} = 0 \qquad T_B = \frac{L_1 J_2}{L_2 J_1} T_A$$

Substitute into the original equilibrium equation,

$$T_A + \frac{L_1 J_2}{L_2 J_1} T_A = 90\,\text{lb}\cdot\text{ft}$$

Sample Problem 3.4



Two solid steel shafts are connected by gears. Knowing that for each shaft $G = 11.2 \times 10^6$ psi and that the allowable shearing stress is 8 ksi, determine (*a*) the largest torque T_0 that may be applied to the end of shaft *AB*, (*b*) the corresponding angle through which end *A* of shaft *AB* rotates.

SOLUTION:

Apply a static equilibrium analysis on the two shafts to find a relationship between T_{CD} and T_0.

Apply a kinematic analysis to relate the angular rotations of the gears.

Find the maximum allowable torque on each shaft – choose the smallest.

Find the corresponding angle of twist for each shaft and the net angular rotation of end *A*.

SOLUTION:

Apply a static equilibrium analysis on the two shafts to find a relationship between T_{CD} and T_0.

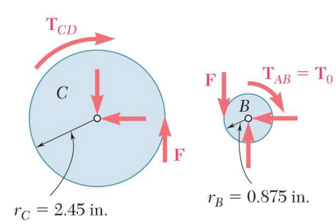


Fig. 1 Free–body diagrams of gears B and C.

$$\sum M_B = 0 = F(0.875\text{in.}) - T_0$$
$$\sum M_C = 0 = F(2.45\text{in.}) - T_{CD}$$
$$T_{CD} = 2.8T_0$$

Apply a kinematic analysis to relate the angular rotations of the gears.

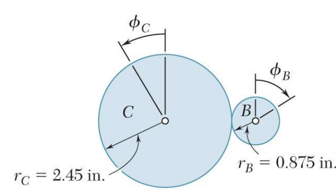


Fig. 2 Angles of twist for gears B and C.

$$r_B\phi_B = r_C\phi_C$$
$$\phi_B = \frac{r_C}{r_B}\phi_C = \frac{2.45\text{in.}}{0.875\text{in.}}\phi_C$$
$$\phi_B = 2.8\phi_C$$

Find the T_0 for the maximum allowable torque on each shaft – choose the smallest.

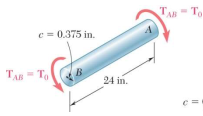


Fig. 3 Free-body diagram of shaft AB.

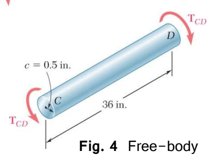


Fig. 4 Free-body diagram of shaft CD.

$$\tau_{\max} = \frac{T_{AB}c}{J_{AB}} \qquad 8000\,psi = \frac{T_0(0.375\,\text{in.})}{\frac{\pi}{2}(0.375\,\text{in.})^4}$$

$$T_0 = 663\,\text{lb}\cdot\text{in.}$$

$$\tau_{\max} = \frac{T_{CD}c}{J_{CD}} \qquad 8000\,psi = \frac{2.8T_0(0.5\,\text{in.})}{\frac{\pi}{2}(0.5\,\text{in.})^4}$$

$$T_0 = 561\,\text{lb}\cdot\text{in.} \qquad \boxed{T_0 = 561\,\text{lb}\cdot\text{in}}$$

Find the corresponding angle of twist for each shaft and the net angular rotation of end *A*.

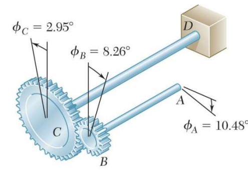


Fig. 5

$$\phi_{A/B} = \frac{T_{AB}L}{J_{AB}G} = \frac{(561\,\text{lb}\cdot\text{in.})(24\,\text{in.})}{\frac{\pi}{2}(0.375\,\text{in.})^4(11.2\times10^6\,\text{psi})}$$

$$= 0.387\,\text{rad} = 2.22^{\circ}$$

$$\phi_{C/D} = \frac{T_{CD}L}{J_{CD}G} = \frac{2.8(561\,\text{lb}\cdot\text{in.})(24\,\text{in.})}{\frac{\pi}{2}(0.5\,\text{in.})^4(11.2\times10^6\,\text{psi})}$$

$$= 0.514\,\text{rad} = 2.95^{\circ}$$

$$\phi_B = 2.8\phi_C = 2.8(2.95^{\circ}) = 8.26^{\circ}$$

$$\phi_A = \phi_B + \phi_{A/B} = 8.26^{\circ} + 2.22^{\circ} \qquad \boxed{\phi_A = 10.48^{\circ}}$$

Design of Transmission Shafts

Principal transmission shaft performance specifications are:

- *power*
- *Speed of rotation*

Designer must select shaft material and dimensions of the cross-section to meet performance specifications without exceeding allowable shearing stress.

Determine torque applied to shaft at specified power and speed,

$$P = T\omega = 2\pi f T$$

$$T = \frac{P}{\omega} = \frac{P}{2\pi f}$$

Find shaft cross-section which will not exceed the maximum allowable shearing stress,

$$\tau_{\max} = \frac{Tc}{J}$$

$$\frac{J}{c} = \frac{\pi}{2}c^3 = \frac{T}{\tau_{\max}} \quad \text{(solid shafts)}$$

$$\frac{J}{c_2} = \frac{\pi}{2c_2}\left(c_2^4 - c_1^4\right) = \frac{T}{\tau_{\max}} \quad \text{(hollow shafts)}$$

Stress Concentrations

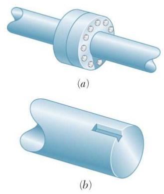


Fig. 3.26 Coupling of shafts using (a) bolted flange, (b) slot for keyway.

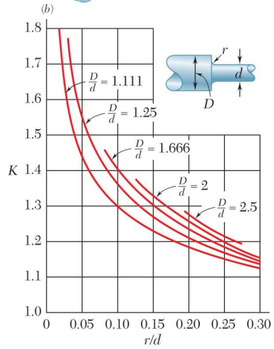


Fig. 3.28 Plot of stress concentration factors for fillets in circular shafts.

The derivation of the torsion formula,

$$\tau_{\max} = \frac{Tc}{J}$$

assumed a circular shaft with uniform cross-section loaded through rigid end plates.

The use of flange couplings, gears and pulleys attached to shafts by keys in keyways, and cross-section discontinuities can cause stress concentrations

Experimental or numerically determined concentration factors are applied as

$$\tau_{\max} = K\frac{Tc}{J}$$