ME210 STRENGTH OF MATERIALS

CHAPTER 2
Stress and Strain – Axial Loading

Stress & Strain: Axial Loading

Suitability of a structure or machine may depend on the deformations in the structure as well as the stresses induced under loading. Statics analyses alone are not sufficient.

Considering structures as deformable allows determination of member forces and reactions which are statically indeterminate.

Determination of the stress distribution within a member also requires consideration of deformations in the member.

Chapter 2 is concerned with deformation of a structural member under axial loading. Later chapters will deal with torsional and pure bending loads.

Normal Strain

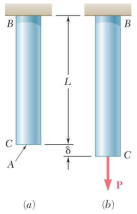


Fig. 2.1 Undeformed and deformed axially loaded rod.

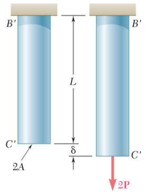


Fig. 2.3 Twice the load is required to obtain the same deformation δ when the cross-sectional area is doubled.

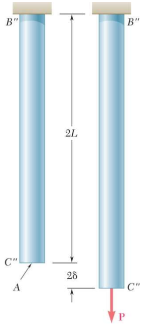


Fig. 2.4 The deformation is doubled when the rod length is doubled while keeping the load **P** and cross-sectional area A.

$$\sigma = \frac{P}{A} = \text{stress}$$
$$\varepsilon = \frac{\delta}{L} = \text{normal strain}$$

$$\sigma = \frac{2P}{2A} = \frac{P}{A}$$
$$\varepsilon = \frac{\delta}{L}$$

$$\sigma = \frac{P}{A}$$
$$\varepsilon = \frac{2\delta}{2L} = \frac{\delta}{L}$$

Stress-Strain Test

Photo 2.2 Universal test machine used to test tensile specimens.

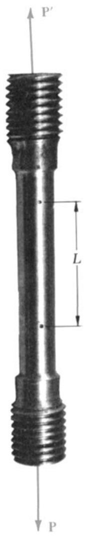
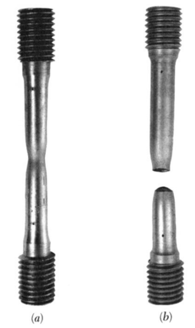


Photo 2.3 Elongated tensile test specimen having load **P** and deformed length $L > L_0$.

Stress-Strain Diagram: Ductile Materials



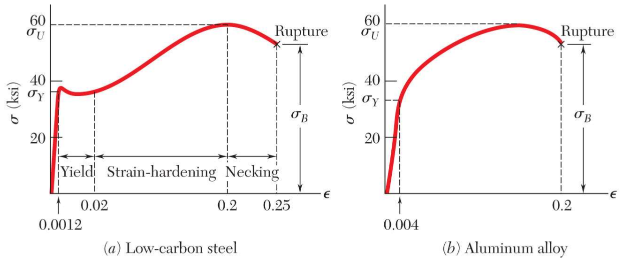


Photo 2.4 Ductile material tested specimens: (a) with cross-section necking, (b) ruptured.

Fig. 2.6 Stress-strain diagrams of two typical ductile materials.

Stress-Strain Diagram: Brittle Materials

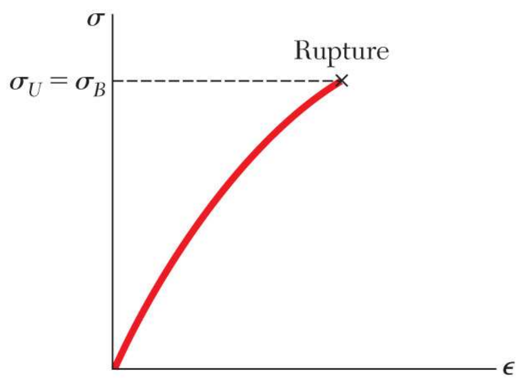


Photo 2.5 Ruptured brittle materials specimen.

Fig 2.7 Stress-strain diagram for a typical brittle material.

Hooke's Law: Modulus of Elasticity

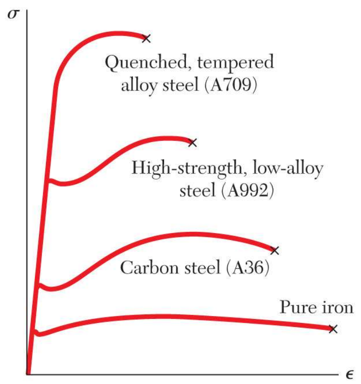


Fig 2.11 Stress-strain diagrams for iron and different grades of steel.

Below the yield stress

$\sigma = E\varepsilon$

E = Youngs Modulus or Modulus of Elasticity

Strength is affected by alloying, heat treating, and manufacturing process but stiffness (Modulus of Elasticity) is not.

Elastic vs. Plastic Behavior

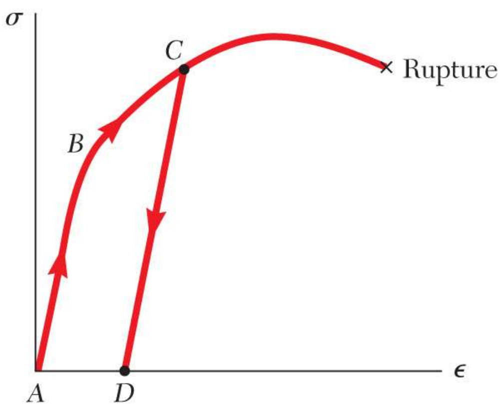


Fig. 2.13 Stress-strain response of ductile material load beyond yield and unloaded.

If the strain disappears when the stress is removed, the material is said to behave *elastically*.

The largest stress for which this occurs is called the *elastic limit*.

When the strain does not return to zero after the stress is removed, *plastic deformation* of the material has taken place.

Fatigue

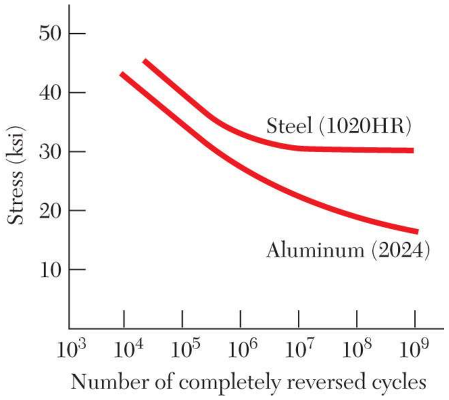


Fig. 2.16 Typical σ-n curves.

Fatigue properties are shown on σ-N diagrams.

A member may fail due to *fatigue* at stress levels significantly below the ultimate strength if subjected to many loading cycles.

When the stress is reduced below the *endurance limit*, fatigue failures do not occur for any number of cycles.

Deformations Under Axial Loading

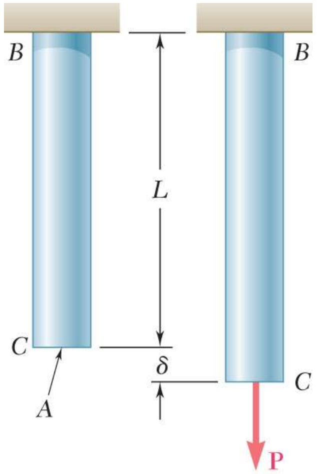


Fig. 2.17 Undeformed and deformed axially-loaded rod.

From Hooke's Law:

$$\sigma = E\varepsilon \qquad \varepsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

From the definition of strain:

$$\varepsilon = \frac{\delta}{L}$$

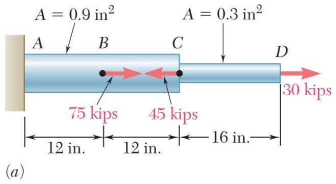
Equating and solving for the deformation,

$$\delta = \frac{PL}{AE}$$

With variations in loading, cross-section or material properties,

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i}$$

Concept Application 2.1



$E = 29 \times 10^{-6}$ psi

Determine the deformation of the steel rod shown under the given loads.

SOLUTION:

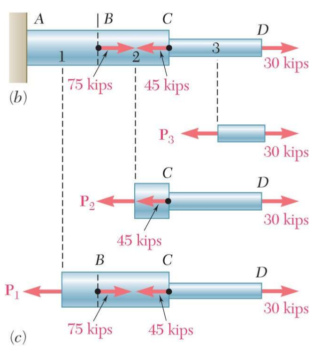
Divide the rod into components at the load application points.

Apply a free-body analysis on each component to determine the internal force

Evaluate the total of the component deflections.

SOLUTION:

Divide the rod into three components:



$L_1 = L_2 = 12\,\text{in.}$ $\quad L_3 = 16\,\text{in.}$

$A_1 = A_2 = 0.9\,\text{in}^2$ $\quad A_3 = 0.3\,\text{in}^2$

Apply free-body analysis to each component to determine internal forces,

$P_1 = 60 \times 10^3\,\text{lb}$

$P_2 = -15 \times 10^3\,\text{lb}$

$P_3 = 30 \times 10^3\,\text{lb}$

Evaluate total deflection,

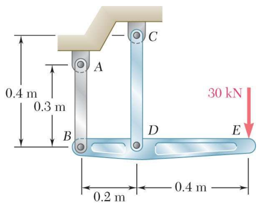
$$\delta = \sum_i \frac{P_i L_i}{A_i E_i} = \frac{1}{E}\left(\frac{P_1 L_1}{A_1} + \frac{P_2 L_2}{A_2} + \frac{P_3 L_3}{A_3}\right)$$

$$= \frac{1}{29 \times 10^6}\left[\frac{\left(60 \times 10^3\right)12}{0.9} + \frac{\left(-15 \times 10^3\right)12}{0.9} + \frac{\left(30 \times 10^3\right)16}{0.3}\right]$$

$$= 75.9 \times 10^{-3}\,\text{in.}$$

$$\boxed{\delta = 75.9 \times 10^{-3}\,\text{in.}}$$

Sample Problem 2.1



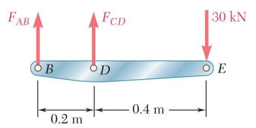
The rigid bar *BDE* is supported by two links *AB* and *CD*.
Link *AB* is made of aluminum (E = 70 GPa) and has a cross-sectional area of 500 mm^2. Link *CD* is made of steel (E = 200 GPa) and has a cross-sectional area of (600 mm^2).
For the 30-kN force shown, determine the deflection (a) of *B*, (b) of *D*, and (c) of *E*.

SOLUTION:

- Apply a free-body analysis to the bar *BDE* to find the forces exerted by links *AB* and *DC*.
- Evaluate the deformation of links *AB* and *DC* or the displacements of *B* and *D*.
- Work out the geometry to find the deflection at E given the deflections at B and D.

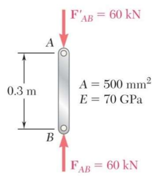
SOLUTION:

Free body: Bar *BDE*



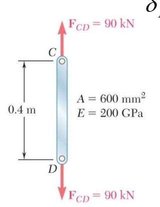
$$+\circlearrowleft \sum M_B = 0$$
$$0 = -(30\,\text{kN})(0.6\,\text{m}) + F_{CD}(0.2\,\text{m})$$
$$F_{CD} = +90\,\text{kN} \quad F_{CD} = 90\,\text{kN}\ \textit{tension}$$
$$+\circlearrowleft \sum M_D = 0$$
$$0 = -(30\,\text{kN} \times 0.4\,\text{m}) - F_{AB} \times 0.2\,\text{m}$$
$$F_{AB} = -60\,\text{kN} \quad F_{AB} = 60\,\text{kN}\ \textit{compression}$$

Displacement of *B*:

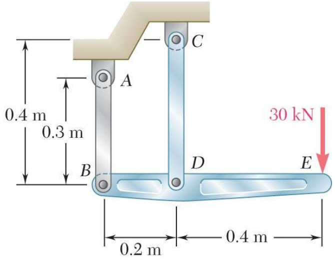


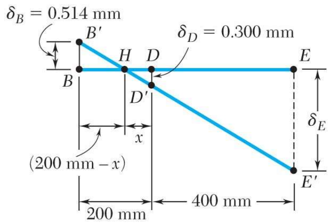
$$\delta_B = \frac{PL}{AE}$$
$$= \frac{\left(-60\times10^{3}\,\text{N}\right)\left(0.3\,\text{m}\right)}{\left(500\times10^{-6}\,\text{m}^2\right)\left(70\times10^{9}\,\text{Pa}\right)}$$
$$= -514\times10^{-6}\,\text{m}$$
$$\boxed{\delta_B = 0.514\,\text{mm}\uparrow}$$

Displacement of *D*:



$$\delta_D = \frac{PL}{AE}$$
$$= \frac{\left(90\times10^{3}\,\text{N}\right)\left(0.4\,\text{m}\right)}{\left(600\times10^{-6}\,\text{m}^2\right)\left(200\times10^{9}\,\text{Pa}\right)}$$
$$= 300\times10^{-6}\,\text{m}$$
$$\boxed{\delta_D = 0.300\,\text{mm}\downarrow}$$





Displacement of D:

$$\frac{BB'}{DD'} = \frac{BH}{HD}$$

$$\frac{0.514\text{ mm}}{0.300\text{ mm}} = \frac{(200\text{ mm}) - x}{x}$$

$$x = 73.7\text{ mm}$$

$$\frac{EE'}{DD'} = \frac{HE}{HD}$$

$$\frac{\delta_E}{0.300\text{ mm}} = \frac{(400 + 73.7)\text{mm}}{73.7\text{ mm}}$$

$$\delta_E = 1.928\text{ mm}$$

$$\delta_E = 1.928\text{ mm} \downarrow$$

Static Indeterminate Problems

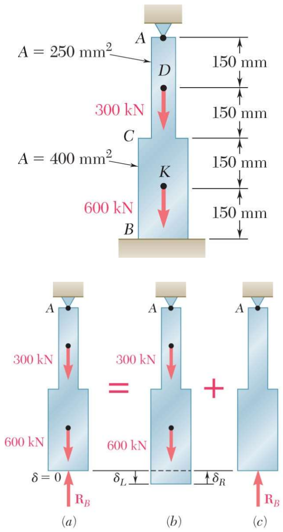


Fig. 2.23

Structures for which internal forces and reactions cannot be determined from statics alone are said to be *statically indeterminate*.

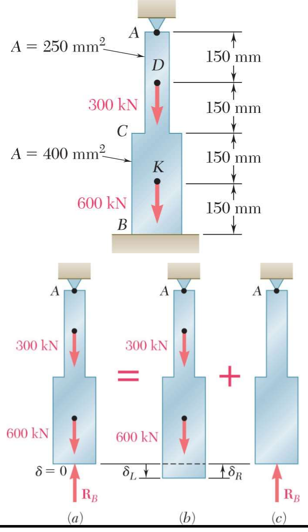
A structure will be statically indeterminate whenever it is held by more supports than are required to maintain its equilibrium.

Redundant reactions are replaced with unknown loads which along with the other loads must produce compatible deformations.

Deformations due to actual loads and redundant reactions are determined separately and then added.

$$\delta = \delta_L + \delta_R = 0$$

Concept Application 2.4



Determine the reactions at A and B for the steel bar and loading shown, assuming a close fit at both supports before the loads are applied.

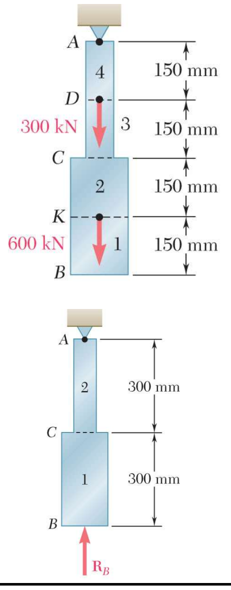
SOLUTION:

Consider the reaction at B as redundant, release the bar from that support, and solve for the displacement at B due to the applied loads.

Solve for the displacement at B due to the redundant reaction at $\mathbf{R}_B$.

Require that the displacements due to the loads and due to the redundant reaction be compatible, i.e., require that their sum be zero.

Solve for the reaction at R_A due to applied loads and the reaction found at R_B.



SOLUTION:

Solve for the displacement at B due to the applied loads with the redundant constraint released,

$$P_1 = 0 \quad P_2 = P_3 = 600\times10^3\,\text{N} \quad P_4 = 900\times10^3\,\text{N}$$

$$A_1 = A_2 = 400\times10^{-6}\,\text{m}^2 \quad A_3 = A_4 = 250\times10^{-6}\,\text{m}^2$$

$$L_1 = L_2 = L_3 = L_4 = 0.150\,\text{m}$$

$$\delta_L = \sum_i \frac{P_i L_i}{A_i E_i} = \frac{1.125\times10^9}{E}$$

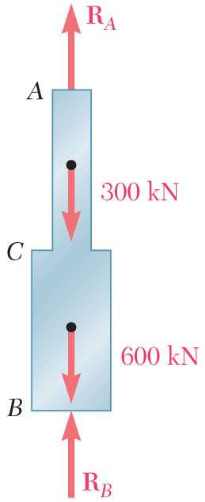
Solve for the displacement at B due to the redundant constraint,

$$P_1 = P_2 = -R_B$$

$$A_1 = 400\times10^{-6}\,\text{m}^2 \quad A_2 = 250\times10^{-6}\,\text{m}^2$$

$$L_1 = L_2 = 0.300\,\text{m}$$

$$\delta_R = \sum_i \frac{P_i L_i}{A_i E_i} = -\frac{\left(1.95\times10^3\right)R_B}{E}$$



Require that the displacements due to the loads and due to the redundant reaction be compatible,

$$\delta = \delta_L + \delta_R = 0$$

$$\delta = \frac{1.125\times10^9}{E} - \frac{\left(1.95\times10^3\right)R_B}{E} = 0$$

$$R_B = 577\times10^3\,\text{N} = 577\,\text{kN}$$

Find the reaction at A due to the loads and the reaction at B

$$+\uparrow\sum F_y = 0 = R_A - 300\,\text{kN} - 600\,\text{kN} + 577\,\text{kN}$$

$$R_A = 323\,\text{kN}$$

$$R_A = 323\,\text{kN}$$
$$R_B = 577\,\text{kN}$$

Problems Involving Temperature Change

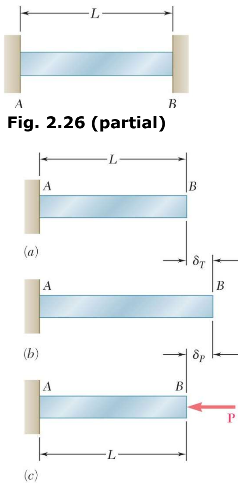


Fig. 2.27 Superposition method to find force at point *B* of restrained rod *AB* undergoing thermal expansion. (a) Initial rod length; (b) thermally expanded rod length; (c) force *P* pushes point *B* back to zero deformation.

A temperature change results in a change in length or *thermal strain*. There is no stress associated with the thermal strain unless the elongation is restrained by the supports.

Treat the additional support as redundant and apply the principle of superposition.

$$\delta_T = \alpha(\Delta T)L \qquad \delta_P = \frac{PL}{AE}$$

$$\alpha = \text{coefficient of thermal expansion}$$

The thermal deformation and the deformation from the redundant support must be compatible.

$$\delta = \delta_T + \delta_P = 0$$

$$\alpha(\Delta T)L + \frac{PL}{AE} = 0$$

$$P = -AE\alpha(\Delta T)$$

$$\sigma = \frac{P}{A} = -E\alpha(\Delta T)$$

Poisson's Ratio

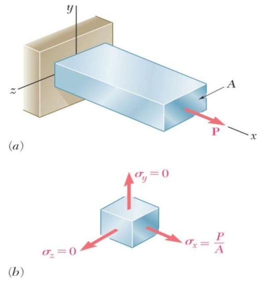


Fig. 2.29 A bar in uniaxial tension and a representative stress element.

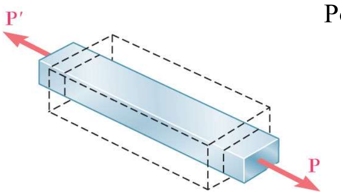


Fig. 2.30 Materials undergo transverse contraction when elongated under axial load.

For a slender bar subjected to axial loading:

$$\varepsilon_x = \frac{\sigma_x}{E} \quad \sigma_y = \sigma_z = 0$$

The elongation in the x-direction is accompanied by a contraction in the other directions. Assuming that the material is *homogeneous* and *isotropic* (no directional dependence),

$$\varepsilon_y = \varepsilon_z \neq 0$$

Poisson's ratio is defined as

$$\nu = -\left|\frac{\text{lateral strain}}{\text{axial strain}}\right| = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x}$$

Multiaxial Loading: Generalized Hooke's Law

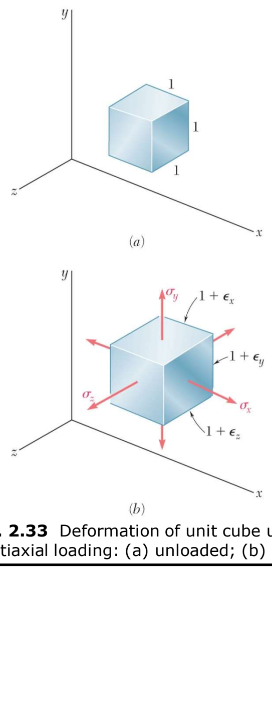


Fig. 2.33 Deformation of unit cube under multiaxial loading: (a) unloaded; (b) deformed.

For an element subjected to multi-axial loading, the normal strain components resulting from the stress components may be determined from the *principle of superposition*. This requires:

1) Each effect is linearly related to the load that produces it.

2) The deformation resulting form any given load is small and does not affect the conditions of application of the other loads.

With these restrictions:

$$\varepsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\varepsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\varepsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

Dilatation: Bulk Modulus

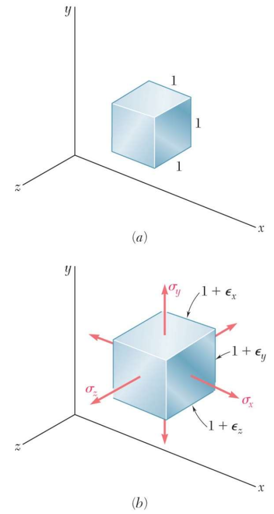


Fig. 2.33 Deformation of unit cube under multiaxial loading: (a) unloaded; (b) deformed.

Relative to the unstressed state, the change in volume is

$$\begin{aligned} e &= \nu - 1 = 1 + \left[\varepsilon_x + \varepsilon_y + \varepsilon_z\right] - 1 \\ &= \varepsilon_x + \varepsilon_y + \varepsilon_z \\ &= \frac{1-2\nu}{E}\left(\sigma_x + \sigma_y + \sigma_z\right) \\ &= \text{dilatation (change in volume per unit volume)} \end{aligned}$$

For element subjected to uniform hydrostatic pressure,

$$e = -\frac{3(1-2\nu)}{E}p = -\frac{p}{k}$$

$$k = \frac{E}{3(1-2\nu)} = \text{bulk modulus or modulus of compression}$$

Subjected to uniform pressure, dilatation must be negative, therefore

$$0 < \nu < \tfrac{1}{2}$$

Shearing Strain

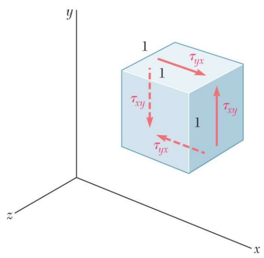


Fig. 2.36 Unit cubic element subjected to shearing stress.

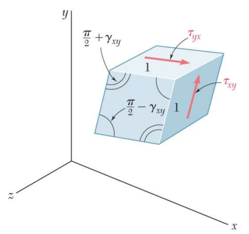


Fig. 2.37 Deformation of unit cubic element due to shearing stress.

A cubic element subjected to only shearing stress will deform into a rhomboid. The corresponding *shearing strain* is quantified in terms of the change in angle between the sides,

$$\tau_{xy} = f\left(\gamma_{xy}\right)$$

A plot of shearing stress vs. shearing strain is similar to the previous plots of normal stress vs. normal strain except that the strength values are approximately half. For values of shearing strain that do not exceed the proportional limit,

$$\tau_{xy} = G\gamma_{xy} \quad \tau_{yz} = G\gamma_{yz} \quad \tau_{zx} = G\gamma_{zx}$$

where G is the *modulus of rigidity* or *shear modulus*.

Concept Application 2.10

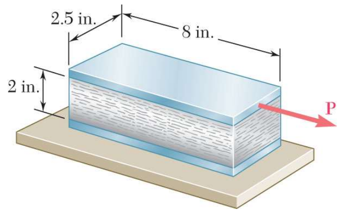


Fig. 2.41(a) Rectangular block loaded in shear.

A rectangular block of material with modulus of rigidity $G = 90$ ksi is bonded to two rigid horizontal plates. The lower plate is fixed, while the upper plate is subjected to a horizontal force **P**. Knowing that the upper plate moves through 0.04 in. under the action of the force, determine (a) the average shearing strain in the material, and (b) the force **P** exerted on the plate.

SOLUTION:

Determine the average angular deformation or shearing strain of the block.

Apply Hooke's law for shearing stress and strain to find the corresponding shearing stress.

Use the definition of shearing stress to find the force **P**.

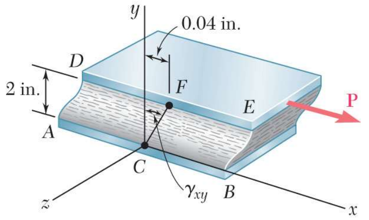


Fig. 2.41(b) Deformed block showing the shear strain.

Determine the average angular deformation or shearing strain of the block.

$$\gamma_{xy} \approx \tan\gamma_{xy} = \frac{0.04\,\text{in.}}{2\,\text{in.}} \qquad \gamma_{xy} = 0.020\,\text{rad}$$

Apply Hooke's law for shearing stress and strain to find the corresponding shearing stress.

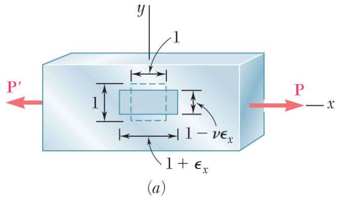
$$\tau_{xy} = G\gamma_{xy} = \left(90\times10^3\,\text{psi}\right)\left(0.020\,\text{rad}\right) = 1800\,\text{psi}$$

Use the definition of shearing stress to find the force **P**.

$$P = \tau_{xy}A = \left(1800\,\text{psi}\right)\left(8\,\text{in.}\right)\left(2.5\,\text{in.}\right) = 36\times10^3\,\text{lb}$$

$$\boxed{P = 36.0\,\text{kips}}$$

Relation Between *E*, *ν*, and *G*



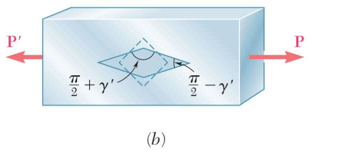


Fig. 2.42 Representations of strain in an axially loaded bar: (a) cubicstrain element faces aligned with coordinate axes; (b) cubicstrain element faces rotated 45º about z-axis.

An axially loaded slender bar will elongate in the *x* direction and contract in both of the transverse *y* and *z* directions.

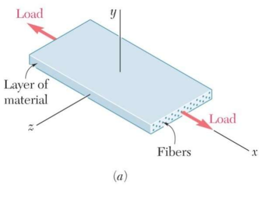
An initially cubic element oriented as in Figure 2.42(a) will deform into a rectangular parallelepiped. The axial load produces a normal strain.

If the cubic element is oriented as in Figure 2.42(b), it will deform into a rhombus. Axial load also results in a shearing strain.

Components of normal and shearing strain are related,

$$\frac{E}{2G}=(1+\nu) \quad \text{or} \quad G=\frac{E}{2(1+\nu)}$$

Composite Materials



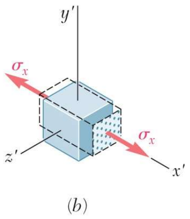


Fig. 2.44 Orthotropic Fiber-reinforced composite material under uniaxial tensile load.

Fiber-reinforced composite materials are fabricated by embedding fibers of a strong, still material into a weaker, softer material called a *matrix*.

Normal stresses and strains are related by Hooke's Law but with directionally dependent moduli of elasticity,

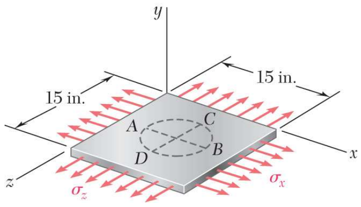
$$E_x=\frac{\sigma_x}{\varepsilon_x} \quad E_y=\frac{\sigma_y}{\varepsilon_y} \quad E_z=\frac{\sigma_z}{\varepsilon_z}$$

Transverse contractions are related by directionally dependent values of Poisson's ratio, e.g.,

$$\nu_{xy}=-\frac{\varepsilon_y}{\varepsilon_x} \quad \nu_{xz}=-\frac{\varepsilon_z}{\varepsilon_x}$$

The three components of strain ε_x, ε_y, and ε_z for orthotropic materials can be expressed in terms of normal stress only and do not depend upon any shearing stresses.

Sample Problem 2.5



A circle of diameter d = 9 in. is scribed on an unstressed aluminum plate of thickness t = 3/4 in. Forces acting in the plane of the plate later cause normal stresses σ_x = 12 ksi and σ_z = 20 ksi.

For $E = 10\text{x}10^6$ psi and ν = 1/3, determine the change in:

a) the length of diameter AB,
b) the length of diameter CD,
c) the thickness of the plate, and
d) the volume of the plate.

SOLUTION:

Apply the generalized Hooke's Law to find the three components of normal strain.

$$\varepsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$= \frac{1}{10\times10^6\,\text{psi}}\left[(12\,\text{ksi}) - 0 - \frac{1}{3}(20\,\text{ksi})\right]$$

$$= +0.533\times10^{-3}\,\text{in./in.}$$

$$\varepsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$= -1.067\times10^{-3}\,\text{in./in.}$$

$$\varepsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

$$= +1.600\times10^{-3}\,\text{in./in.}$$

Evaluate the deformation components.

$$\delta_{B/A} = \varepsilon_x d = (+0.533\times10^{-3}\,\text{in./in.})(9\,\text{in.})$$

$$\boxed{\delta_{B/A} = +4.8\times10^{-3}\,\text{in.}}$$

$$\delta_{C/D} = \varepsilon_z d = (+1.600\times10^{-3}\,\text{in./in.})(9\,\text{in.})$$

$$\boxed{\delta_{C/D} = +14.4\times10^{-3}\,\text{in.}}$$

$$\delta_t = \varepsilon_y t = (-1.067\times10^{-3}\,\text{in./in.})(0.75\,\text{in.})$$

$$\boxed{\delta_t = -0.800\times10^{-3}\,\text{in.}}$$

Find the change in volume

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = 1.067\times10^{-3}\,\text{in}^3/\text{in}^3$$

$$\Delta V = eV = 1.067\times10^{-3}(15\times15\times0.75)\,\text{in}^3$$

$$\boxed{\Delta V = +0.187\,\text{in}^3}$$

Saint-Venant's Principle

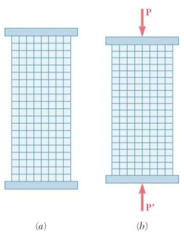


Fig. 2.47 Axial load applied by rigid plates to rubber model.

Fig. 2.48 Concentrated axial load applied to rubber model.

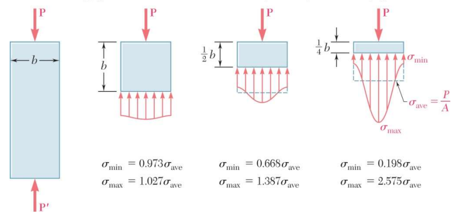


Fig. 2.49 Stress distributions in a plate under concentrated axial loads.

Loads transmitted through rigid plates result in uniform distribution of stress and strain.

Concentrated loads result in large stresses in the vicinity of the load application point.

Stress and strain distributions become uniform at a relatively short distance from the load application points.

Saint-Venant's Principle:
Stress distribution may be assumed independent of the mode of load application except in the immediate vicinity of load application points.

Stress Concentration: Hole

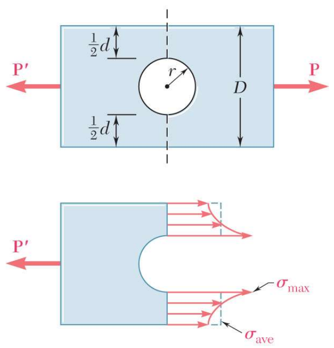


Fig. 2.50 Stress distribution near circular hole in flat bar under axial loading.

Stress concentration factor

$$K = \frac{\sigma_{max}}{\sigma_{ave}}$$

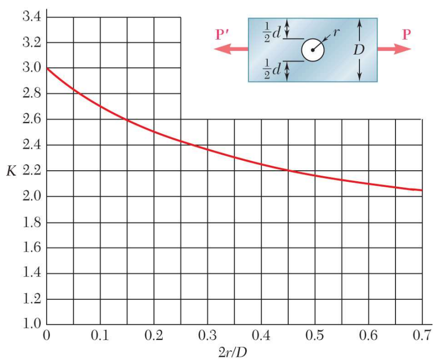


Fig. 2.52(a) Stress concentration factors for flat bars under axial loading.

Discontinuities of cross section may result in high localized or *concentrated* stresses.

Stress Concentration: Fillet

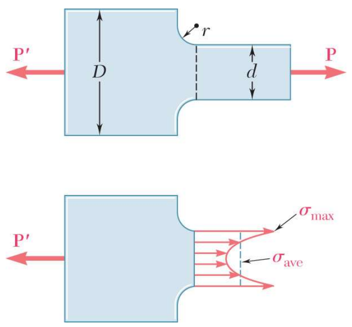


Fig. 2.51 Stress distribution near fillets in flat bar under axial loading.

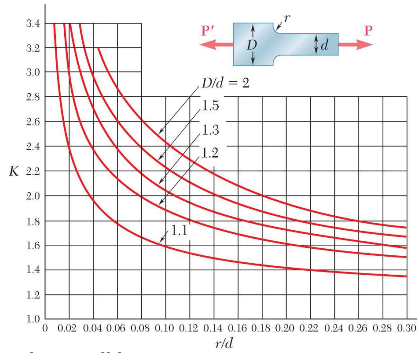
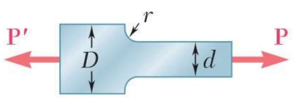


Fig. 2.52(b) Stress concentration factors for flat bars under axial loading.

Concept Application 2.12



Determine the largest axial load **P** that can be safely supported by a flat steel bar consisting of two portions, both 10 mm thick, and respectively 40 and 60 mm wide, connected by fillets of radius r = 8 mm. Assume an allowable normal stress of 165 MPa.

SOLUTION:

Determine the geometric ratios and find the stress concentration factor from Figure 2.52.

Find the allowable average normal stress using the material allowable normal stress and the stress concentration factor.

Apply the definition of normal stress to find the allowable load.

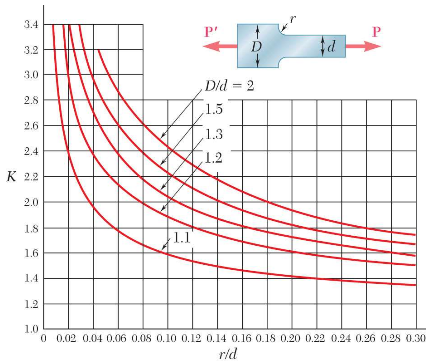


Fig. 2.52(b) Stress concentration factors for flat bars under axial loading.

Determine the geometric ratios and find the stress concentration factor from Figure 2.52.

$$\frac{D}{d} = \frac{60\,\text{mm}}{40\,\text{mm}} = 1.50 \qquad \frac{r}{d} = \frac{8\,\text{mm}}{40\,\text{mm}} = 0.20$$

$$K = 1.82$$

Find the allowable average normal stress using the material allowable normal stress and the stress concentration factor.

$$\sigma_{\text{ave}} = \frac{\sigma_{\text{max}}}{K} = \frac{165\,\text{MPa}}{1.82} = 90.7\,\text{MPa}$$

Apply the definition of normal stress to find the allowable load.

$$P = A\sigma_{ave} = (40\,\text{mm})(10\,\text{mm})(90.7\,\text{MPa})$$

$$= 36.3\times10^{3}\,\text{N}$$

$$\boxed{P = 36.3\,\text{kN}}$$