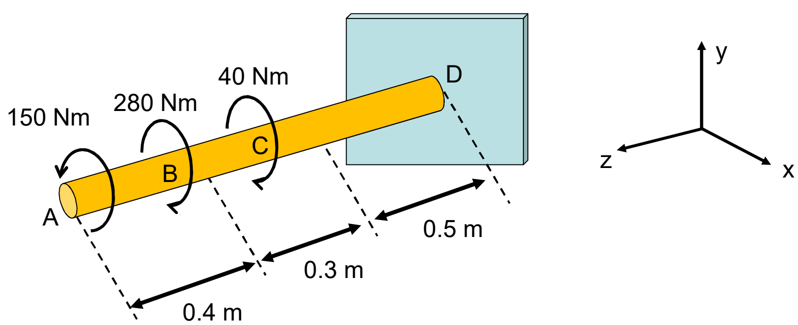
ME210 STRENGTH OF MATERIALS

Chapter 03 – Recitation 2

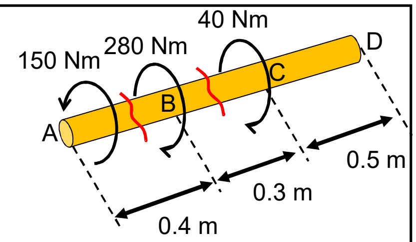
Example 1

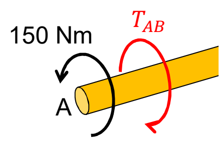


The rod has a diameter of 14 mm and is subjected to external torques as shown in the figure. Determine the angle of twist at A. (G =80 Gpa)

Example 1

- We should find the internal torques
- We can approach by taking internal cuts and drawing free body diagrams

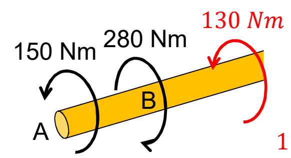


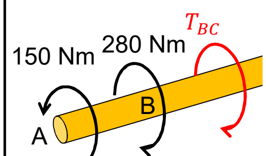


$\Sigma M_z = 0$

$150 - T_{AB} = 0$

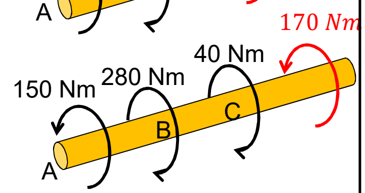
$T_{AB} = +150Nm$

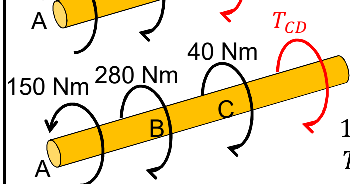




$150 - 280 \ - T_{BC} = 0$

$T_{BC} = -130Nm$





$150 - 280 \ - 40 - T_{CD} = 0$

$T_{CD} = -170Nm$

Example 1

$$J = \frac{1}{2}\pi r^4 = \frac{1}{2}\pi(0.007m)^4 = 3.771 \ * 10^{-9}m^4$$

$T_{AB} = +150Nm$
$T_{BC} = -130Nm$
$T_{CD} = -170Nm$

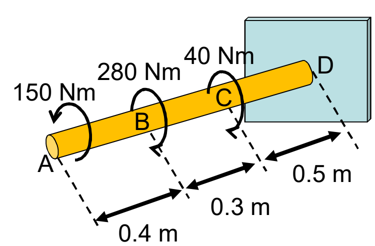
- Summation of angle of twists:

$$\phi_A = \phi_{A/B} + \phi_{B/C} + \phi_{C/D} = \phi_{A/D}$$

$$\phi_A = \sum_i \frac{T_i L_i}{JG} = +\frac{(150)(0.4)}{(3.771 * 10^{-9})(80 * 10^9)}$$

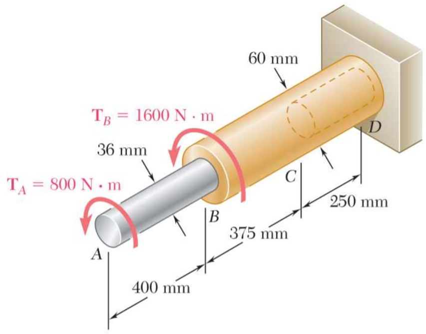
$$-\frac{(130)(0.3)}{(3.771 * 10^{-9})(80 * 10^9)}$$

$$-\frac{(170)(0.5)}{(3.771 * 10^{-9})(80 * 10^9)}$$



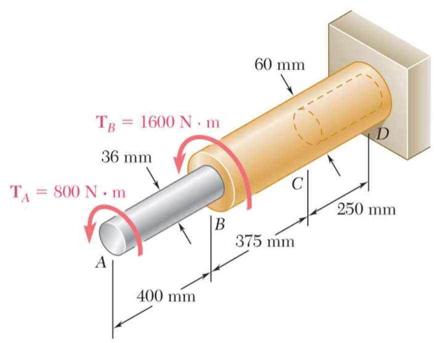
$\phi_A = -0.2121$ rad

Example 2



The aluminum rod AB (G = 27 GPa) is bonded to the brass rod BD (G = 39 GPa). Knowing that portion CD of the brass rod is hollow and has an inner diameter of 40 mm, determine the angle of twist at A.

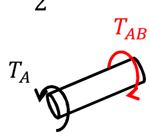
Example 2



- Rod AB

$G = 27 * 10^9 \ Pa \qquad L = 0.400 \ m$

$c = \frac{1}{2} d = 0.018 m$



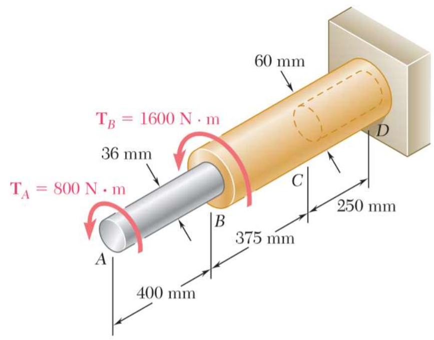
$T_A - T_{AB} = 0$

$T_{AB} = 800 \ Nm$

$$J = \frac{\pi}{2} c^4 = \frac{\pi}{2} (0.018)^4 = 164.896 \ * 10^{-9} \ m^4$$

$$\phi_{A/B} = \frac{TL}{GJ} = \frac{(800)(0.400)}{(27 * 10^9)(164.896 * 10^{-9})} = 71.875 * 10^{-3} rad$$

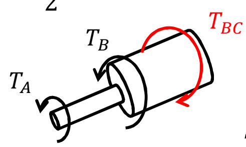
Example 2



- Part BC

$G = 39 * 10^9\ Pa \qquad L = 0.375\ m$

$c = \frac{1}{2}d = 0.030m$



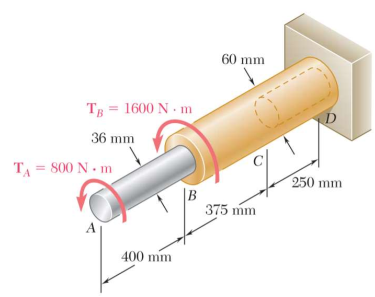
$T_A + T_B - T_{BC} = 0$

$T_{BC} = 800 + 1600 = 2400\ Nm$

$J = \frac{\pi}{2}c^4 = \frac{\pi}{2}(0.030)^4 = 1.27234 * 10^{-6}\ m^4$

$$\phi_{B/C} = \frac{TL}{GJ} = \frac{(2400)(0.375)}{(39 * 10^9)(1.27234 * 10^{-6})} = 18.137 * 10^{-3} rad$$

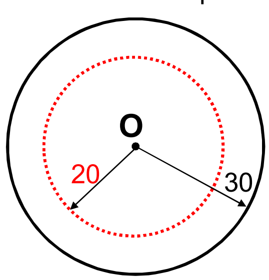
Example 2



- Part CD

$G = 39 * 10^9\ Pa \qquad L = 0.250\ m$

$T_{BC} = T_{CD}$ (Since there is no external torque at C)



$c_1 = \frac{1}{2}d = 0.020m$

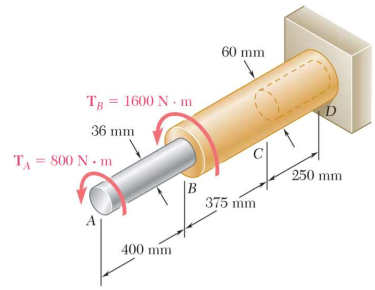
$c_2 = \frac{1}{2}d = 0.030m$

$J = \frac{\pi}{2}(c_2^4 - c_1^4) = \frac{\pi}{2}(0.030^4 - 0.020^4)$

$J = 1.02102 * 10^{-6}\ m^4$

$$\phi_{C/D} = \frac{TL}{GJ} = \frac{(2400)(0.250)}{(39 * 10^9)(1.02102 * 10^{-6})} = 15.068 * 10^{-3} rad$$

Example 2



- Angle of twist at A

$$\phi_{A/B} = 71.875 * 10^{-3} rad$$

$$\phi_{B/C} = 18.137 * 10^{-3} rad$$

$$\phi_{C/D} = 15.068 * 10^{-3} rad$$

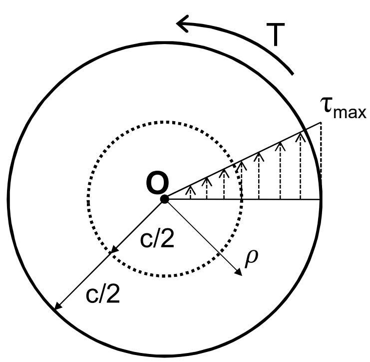
$$\phi_A = \phi_{A/B} + \phi_{B/C} + \phi_{C/D}$$

$$\phi_A = 105.080 * 10^{-3} rad$$

Example 3

The solid shaft of radius 'c' is subjected to a torque T. Determine the fraction of T that is resisted by the material contained within the outer region of the shaft, i.e. $\frac{c}{2} \leq \rho \leq c$

- First draw the problem figure



Shear stress due to torque T is given by,

$$\tau = \frac{T\rho}{J} \qquad \tau = 0 \text{ when } r = 0$$

$$\tau = \frac{Tc}{J} = \tau_{max} \text{ when } r = c$$

or $\tau = \frac{\rho}{c}\tau_{max}$

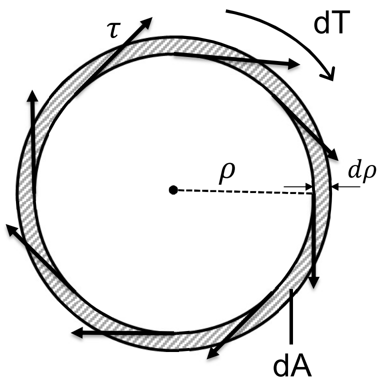
Shear stress increases linearly from origin to outer radius

- We want to find the torque carried by outer region. We can use stress to find force.

Example 3

The solid shaft of radius 'c' is subjected to a torque T. Determine the fraction of T that is resisted by the material contained within the outer region of the shaft, i.e. $\frac{c}{2} \le \rho \le c$

- Imagine an infinitesimal circle with radius ρ, thickness $d\rho$, and area dA



Then the torque dT on this circle can be written as,

$$\underbrace{\rho}_{\text{moment arm}}\ \underbrace{\tau\, dA}_{\text{force}} = \underbrace{dT}_{\text{torque}} \qquad dA = 2\pi\rho d\rho$$

If we take integral of this and apply it to solid shaft, we can find total torque T

$$\int_0^c \rho\,\tau\, dA = \int_0^c dT = T$$

If we take integral from c/2 to c, then we find torque on outer region T'

Example 3

The solid shaft of radius 'c' is subjected to a torque T. Determine the fraction of T that is resisted by the material contained within the outer region of the shaft, i.e. $\frac{c}{2} \le \rho \le c$

$$\int_0^c dT = T \qquad \int_{c/2}^c dT = T' \qquad dT = \rho\,\tau\, dA \qquad dA = 2\pi\rho d\rho$$

$$\tau = \frac{\rho}{c}\tau_{max}$$

$$dT = \rho\frac{\rho}{c}\tau_{max}2\pi\rho d\rho = \frac{2\pi\tau_{max}}{c}\rho^3 d\rho$$

$$T = \int_0^c \frac{2\pi\tau_{max}}{c}\rho^3 d\rho = \frac{2\pi\tau_{max}}{c}\frac{\rho^4}{4}\Big|_0^c \qquad T = \frac{\pi\tau_{max}c^3}{2}$$

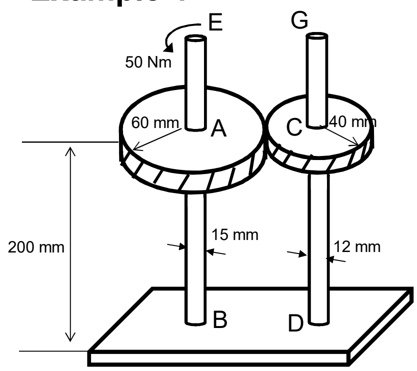
$$T' = \int_{c/2}^c \frac{2\pi\tau_{max}}{c}\rho^3 d\rho = \frac{2\pi\tau_{max}}{c}\frac{\rho^4}{4}\Big|_{c/2}^c \qquad T' = \frac{\pi\tau_{max}c^3}{2} - \frac{\pi\tau_{max}c^3}{32} = \frac{15\pi\tau_{max}c^3}{32}$$

Then the fraction of T resisted by outer region is:

$$\frac{T'}{T} = \frac{\frac{15\pi\tau_{max}c^3}{32}}{\frac{\pi\tau_{max}c^3}{2}} = \mathbf{\frac{15}{16}}$$

Or **93.75%** !!

Example 4



Rotation is prevented at B and D.
A 50 Nm torque is applied at E of shaft EAB.

$r_{AB} = 7.5\ mm \quad r_1 = 60\ mm \quad G = 77\ GPa$

$r_{CD} = 6\ mm \quad r_2 = 40\ mm$

Find,

a) maximum shearing stress in CD
b) the angle of rotation at A

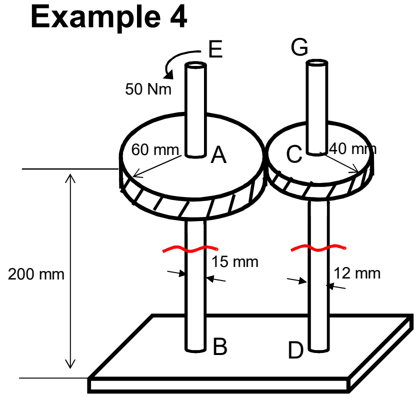
- To calculate these we need to find torque on shaft AB and CD

Let,
Torque in AB = T_{AB}
Torque in CD = T_{CD}
Torque applied at E = T_E

Then,
-Shear stress $\tau = \frac{Tr}{J}$ $\qquad \tau_{CD,max} = \frac{T_{CD} r_{CD}}{J_{CD}}$

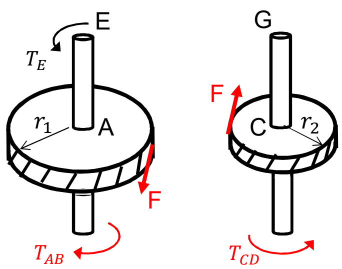
-Angle of twist $\phi = \frac{TL}{GJ}$ $\qquad \phi_A = \frac{T_{AB} L_{AB}}{G J_{AB}}$

Example 4



$\tau_{CD,max} = \frac{T_{CD} r_{CD}}{J_{CD}} \qquad \phi_A = \frac{T_{AB} L_{AB}}{G J_{AB}}$

- We need to find T_{AB} and T_{CD}
- Draw free body diagrams EAB and GCD
- Take cuts at AB and CD



- Static equilibrium

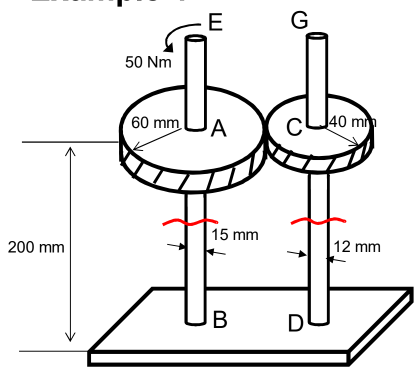
$\sum M_A = 0\ (CCW +) \quad T_E - T_{AB} - F r_1 = 0$

$\sum M_C = 0\ (CCW +) \quad T_{CD} - F r_2 = 0$

$F = \frac{T_{CD}}{r_2} \qquad T_{CD} = \frac{r_2}{r_1}(T_E - T_{AB})$

We need another equation

Example 4



$$T_{CD} = \frac{r_2}{r_1}(T_E - T_{AB}) \quad (1)$$

- We can use kinematics
- When disk A and C rotate, the points on their edges will move by the same amount
- If disk A rotates by ϕ_A, and disk C rotates by ϕ_C, then,

$$\phi_A r_1 = \phi_C r_2 \text{ or } \phi_A 60 = \phi_C 40$$

$$\phi_A = \frac{2}{3}\phi_C \quad (2)$$

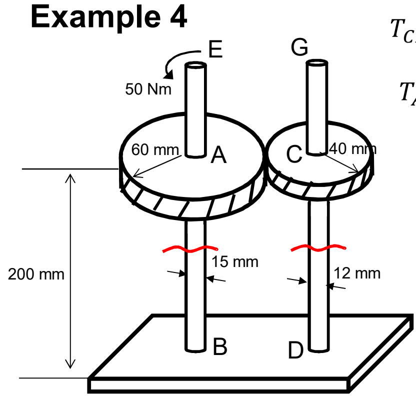
We can write angle of twist as, $\phi_A = \phi_{A/B} = \dfrac{T_{AB}L_{AB}}{GJ_{AB}}$ $\quad \phi_C = \phi_{C/D} = \dfrac{T_{CD}L_{CD}}{GJ_{CD}}$

Using (2), $\dfrac{T_{AB}L_{AB}}{GJ_{AB}} = \dfrac{2}{3}\dfrac{T_{CD}L_{CD}}{GJ_{CD}}$

$$L_{AB} = L_{CD} = L$$

$$T_{AB} = \frac{2}{3}T_{CD}\frac{J_{AB}}{J_{CD}} \quad (3)$$

Example 4



$$T_{CD} = \frac{r_2}{r_1}(T_E - T_{AB}) \quad (1)$$

$$T_{AB} = \frac{2}{3}T_{CD}\frac{J_{AB}}{J_{CD}} \quad (3)$$

- 2 equations 2 unknowns
- J for circular shaft:

$$J_{AB} = \frac{1}{2}\pi r_{AB}^4 = \frac{1}{2}\pi(0.0075)^4 = 4.97x10^{-9}$$

$$J_{CD} = \frac{1}{2}\pi r_{CD}^4 = \frac{1}{2}\pi(0.006)^4 = 2.04x10^{-9}$$

$$r_1 = 0.060\ m, \quad r_2 = 0.040\ m \quad T_E = 50\ Nm$$

$$r_{AB} = 0.0075\ m, \quad r_{CD} = 0.006\ m$$

- Substitute to (1) and (3),

$$L = 0.2\ m \quad G = 77\ GPa$$

$$T_{AB} = \mathbf{26.02}\ Nm \quad T_{CD} = \mathbf{15.99}\ Nm$$

a) maximum shearing stress in CD

$$\tau_{CD,max} = \frac{T_{CD}r_{CD}}{J_{CD}} = \frac{(15.99)(0.006)}{(2.04x10^{-9})}$$

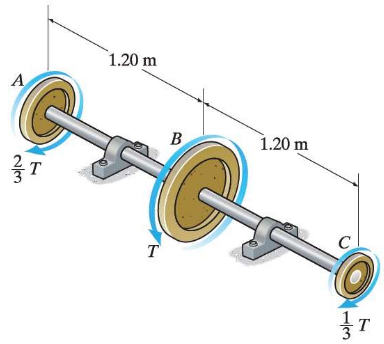
$$\tau_{CD,max} = \mathbf{47.1\ MPa}$$

b) the angle of rotation at A

$$\phi_A = \phi_{A/B} = \frac{T_{AB}L_{AB}}{GJ_{AB}} = \frac{(26.02)(0.2)}{(77x10^9)(4.97x10^{-9})}$$

$$\phi_A = 0.0136\ rad = 0.0136\frac{180}{\pi} = \mathbf{0.779\ degrees}$$

Example 5



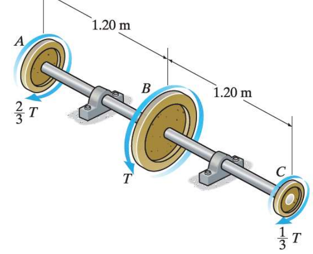
***5–52.** The 60-mm-diameter shaft is made of 6061-T6 aluminum. If the allowable shear stress is $\tau_{allow} = 80$ MPa, and the angle of twist of disk A relative to disk C is limited so that it does not exceed 0.06 rad, determine the maximum allowable torque **T**.

There are 2 restrictions on T

1) $\tau_{allow} = 80\ MPa$
2) $\phi_{A/C} = 0.06\ rad$

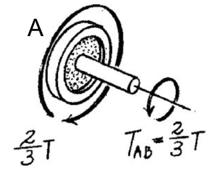
- We need to find internal torques to calculate these. Draw free body at AB and BC

Example 5



- Assume a '+' direction and take internal torques in that direction

Cut AB

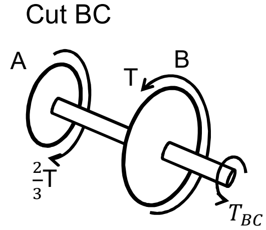


$$T_{AB} - \frac{2}{3}T = 0$$

$$T_{AB} = \frac{2}{3}T$$

- It is easier to use the same half when you take cuts. This makes it easier to follow sign convention

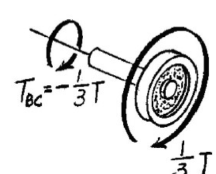
Cut BC



$$T_{BC} - \frac{2}{3}T + T = 0$$

$$T_{BC} = -\frac{1}{3}T$$

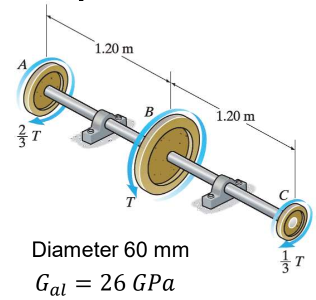
Cut BC (alternative)



$$-T_{BC} - \frac{1}{3}T = 0$$

- Notice that T_{BC} changed direction

Example 5



Diameter 60 mm

$G_{al} = 26\ GPa$

- We found internal torques

$$T_{BC} = -\frac{1}{3}T \qquad T_{AB} = \frac{2}{3}T \qquad (+)$$

- Now apply restrictions

1) $\tau_{allow} = 80\ MPa$

$$\tau_{\text{allow}} = \frac{T_{AB}\,c}{J}$$

T_{AB} is critical since it has bigger magnitude

$$J = \frac{1}{2}\pi c^4 = \frac{1}{2}\pi(0.03^4) = 0.405(10^{-6})\pi\ \text{m}^4$$

$$80(10^3) = \frac{(\frac{2}{3}T)(0.03)}{0.405(10^{-6})\pi} \qquad T = 5089.38\ \text{N}\cdot\text{m} = \underline{5.089\ \text{kN}\cdot\text{m}}$$

2) $\phi_{A/C} = 0.06\ rad$

$$\phi_{A/C} = \Sigma\frac{T_iL_i}{J_iG_i} = \frac{T_{AB}L_{AB}}{JG_{al}} + \frac{T_{BC}L_{BC}}{JG_{al}}$$

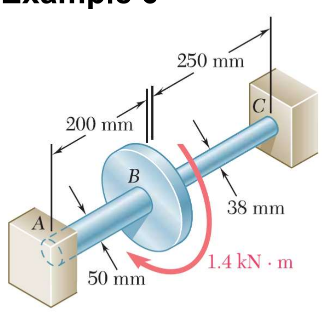
$$0.06 = \frac{(\frac{2}{3}T)(1.2)}{0.405(10^{-6})\pi(26)(10^9)} + \frac{(-\frac{1}{3}T)(1.2)}{0.405(10^{-6})\pi(26)(10^9)}$$

$$T = 4962.14\ \text{N}\cdot\text{m} = \underline{4.96\ \text{kN}\cdot\text{m}}$$

- We take the smaller T as allowable

$$T_{allow} = 4.96\ kN.m$$

Example 6

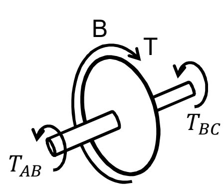


PROBLEM 3.56

Solve Prob. 3.55, assuming that the shaft *AB* is replaced by a hollow shaft of the same outer diameter and 25-mm inner diameter.

PROBLEM 3.55 Two solid steel shafts ($G = 77.2$ GPa) are connected to a coupling disk *B* and to fixed supports at *A* and *C*. For the loading shown, determine (*a*) the reaction at each support, (*b*) the maximum shearing stress in shaft *AB*, (*c*) the maximum shearing stress in shaft *BC*.

- We need internal torques
- This problem is statically indeterminate
- 2 unknown reactions, 1 torque equilibrium



$$T = T_{AB} + T_{BC}$$

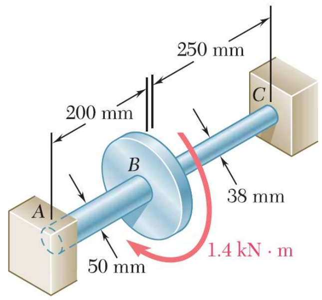
For the extra equation, we can use the fact that A and C are fixed supports. Angle of twists for shaft AB and BC are given by,

$$\phi_{B/A} = \frac{T_{AB}L_{AB}}{J_{AB}G} = \phi_B - \phi_A = \phi_B$$

$$\phi_{B/C} = \frac{T_{BC}L_{BC}}{J_{BC}G} = \phi_B - \phi_C = \phi_B$$

$$\frac{T_{AB}L_{AB}}{J_{AB}G} = \frac{T_{BC}L_{BC}}{J_{BC}G} = \phi_B$$

Example 6



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$$\frac{T_{AB}L_{AB}}{J_{AB}G} = \frac{T_{BC}L_{BC}}{J_{BC}G} = \phi_B$$

Shaft AB: (hollow)

$L_{AB} = 0.200$ m,

$c_2 = 25$ mm $= 0.025$ m

$c_1 = 12.5$ mm $= 0.0125$ m

Shaft BC: (solid)

$L_{BC} = 0.250$ m,

$c = \frac{1}{2}d = 19$ mm $= 0.019$ m

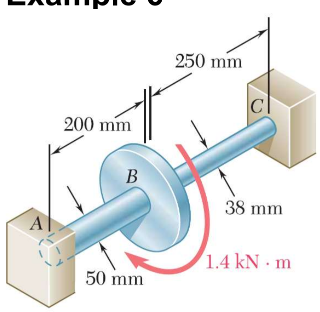
$$J_{AB} = \frac{\pi}{2}\left(c_2^4 - c_1^4\right) = \frac{\pi}{2}(0.025^4 - 0.0125^4) = 575.24 \times 10^{-9}\ \text{m}^4$$

$$T_{AB} = \frac{GJ_{AB}}{L_{AB}}\varphi_B = \frac{(77.2 \times 10^9)(575.24 \times 10^{-9})}{0.200}\varphi_B = 222.04 \times 10^3\,\varphi_B$$

$$J_{BC} = \frac{\pi}{2}c^4 = \frac{\pi}{2}(0.019)^4 = 204.71 \times 10^{-9}\ \text{m}$$

$$T_{BC} = \frac{GJ_{BC}}{L_{BC}}\varphi_B = \frac{(77.2 \times 10^9)(204.71 \times 10^{-9})}{0.250}\varphi_B = 63.214 \times 10^3\varphi_B$$

Example 6



$$T = T_{AB} + T_{BC}$$

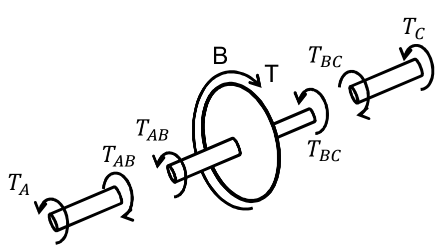
$$1.4 \times 10^3 = 222.04 \times 10^3\varphi_B + 63.214 \times 10^3\varphi_B$$

$$\varphi_B = 4.9079 \times 10^{-3}\ \text{rad}$$

$$T_{AB} = (222.04 \times 10^3)(4.9079 \times 10^{-3}) = 1.08975 \times 10^3\ \text{N} \cdot \text{m}$$

$$T_{BC} = (63.214 \times 10^3)(4.9079 \times 10^{-3}) = 310.25\ \text{N} \cdot \text{m}$$

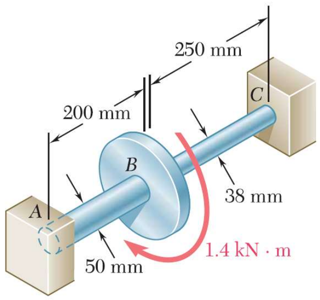
Now, the reactions at the supports can be easily found from T_{AB} *and* T_{BC}



$T_A = T_{AB} = 1090$ N · m ◀

$T_C = T_{BC} = 310$ N · m ◀

Example 6



PROBLEM 3.56

Solve Prob. 3.55, assuming that the shaft AB is replaced by a hollow shaft of the same outer diameter and 25-mm inner diameter.

PROBLEM 3.55 Two solid steel shafts (G = 77.2 GPa) are connected to a coupling disk B and to fixed supports at A and C. For the loading shown, determine (*a*) the reaction at each support, (*b*) the maximum shearing stress in shaft AB, (*c*) the maximum shearing stress in shaft BC.

$$T_A = T_{AB} = 1090 \text{ N} \cdot \text{m} \blacktriangleleft$$

$$T_C = T_{BC} = 310 \text{ N} \cdot \text{m} \blacktriangleleft$$

(*b*) Maximum shearing stress in AB:

$$\tau_{AB} = \frac{T_{AB} c_2}{J_{AB}} = \frac{(1.08975 \times 10^3)(0.025)}{575.243 \times 10^{-9}} = 47.4 \times 10^6 \text{ Pa} \qquad \tau_{AB} = 47.4 \text{ MPa} \blacktriangleleft$$

(*c*) Maximum shearing stress in BC:

$$\tau_{BC} = \frac{T_{BC} c}{J_{BC}} = \frac{(310.25)(0.019)}{204.71 \times 10^{-9}} = 28.8 \times 10^6 \text{ Pa} \qquad \tau_{BC} = 28.8 \text{ MPa} \blacktriangleleft$$