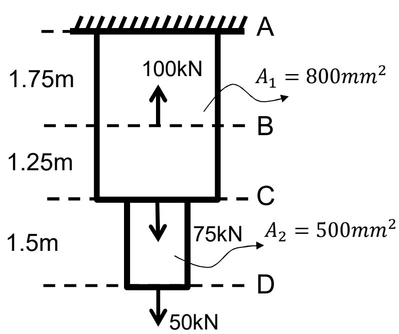
ME210 STRENGTH OF MATERIALS

Chapter 02 – Recitation 1

Example 1



The rod ABCD is made of aluminum alloy with E=70 Gpa. For the loading shown determine the deflection at point D.

- We will analyze the forces at three different sections: **AB, BC, CD**
- Deflection at D is equal to summation of deflections at AB, BC and CD

$$\delta_D = \delta_{AB} + \delta_{BC} + \delta_{CD}$$

- Remember deflection equation for axial loading:

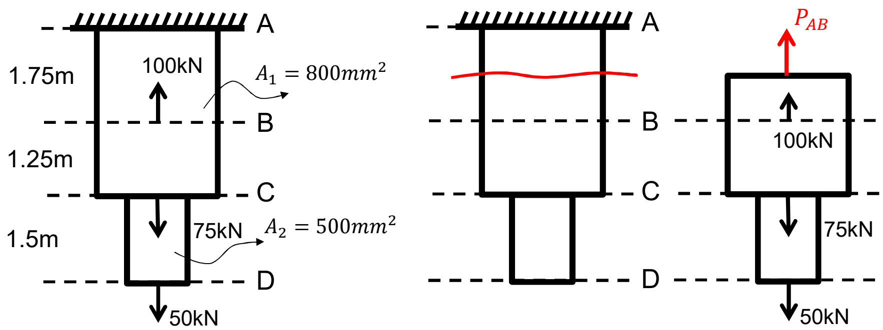
$$\delta = \frac{PL}{AE}$$

- L, A, E are known
- P unknown. It is different for AB, BC, CD sections

- We can take cuts at AB, BC, CD sections and draw free body diagrams to find P

Example 1

- Take cut between AB

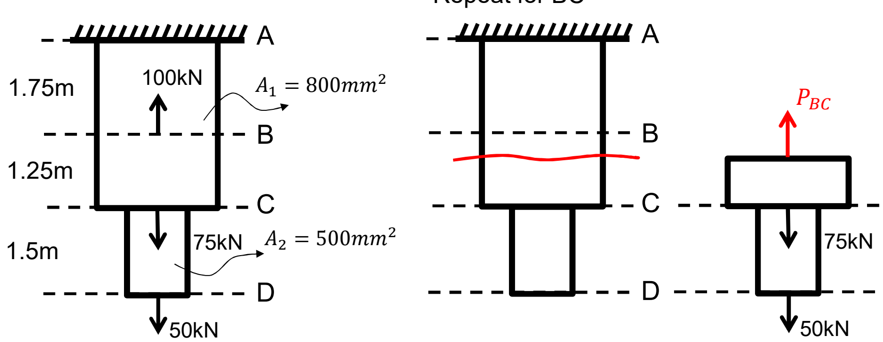


- For the direction of P_{AB}, assume it is **tension (+)**
- Loading is static, total force should be zero.

$P_{AB} + 100 - 75 - 50 = 0$ $\qquad$ $\boldsymbol{P_{AB} = +25\ kN}$ **(tension)**

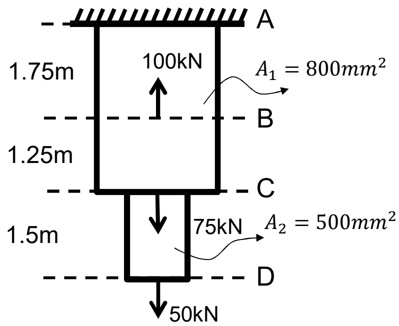
Example 1

- Repeat for BC

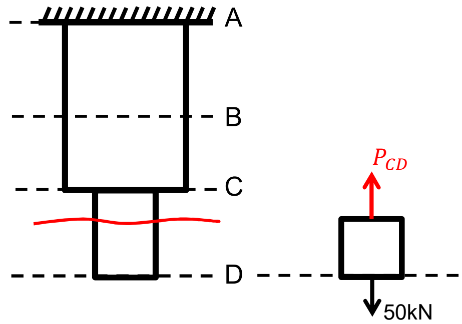


$P_{BC} - 75 - 50 = 0$ $\qquad$ $\boldsymbol{P_{BC} = +125\ kN}$ **(tension)**

Example 1



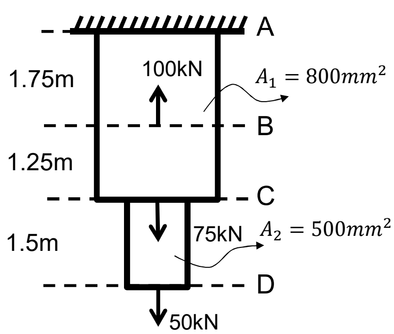
- Repeat for CD



$P_{CD} - 50 = 0$

$\boldsymbol{P_{CD} = +50\ kN}$ **(tension)**

Example 1



The rod ABCD is made of aluminum alloy with E=70 Gpa. For the loading shown determine the deflection at point D.

- Find deflections. We have:

	P [N]	L [m]	A [m^2]	$\delta = \frac{PL}{AE}$ [m]
AB	$25x10^3$	1.75	$800x10^{-6}$	$7.81x10^{-4}$
BC	$125x10^3$	1.25	$800x10^{-6}$	0.0028
CD	$50x10^3$	1.5	$500x10^{-6}$	0.002143

$$\delta_D = \delta_{AB} + \delta_{BC} + \delta_{CD}$$

$$E = 70x10^9 [\frac{N}{m^2}]$$

$\delta_{AB} = 0.781\ mm\ (T)$

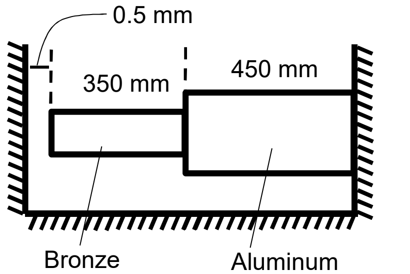
$\delta_{BC} = 2.8\ mm\ (T)$

$\delta_{CD} = 2.143\ mm\ (T)$

- All tension, so we sum them

$$\delta_D = 0.781 + 2.8 + 2.143 = \mathbf{5.724\ mm}$$

Example 2



$A_b = 1500\ mm^2$ $A_{al} = 1800\ mm^2$

$E_b = 105\ GPa$ $E_{al} = 73\ GPa$

$\alpha_b = 21.6x10^{-6}\ C^{-1}$ $\alpha_{al} = 23.2x10^{-6}\ C^{-1}$

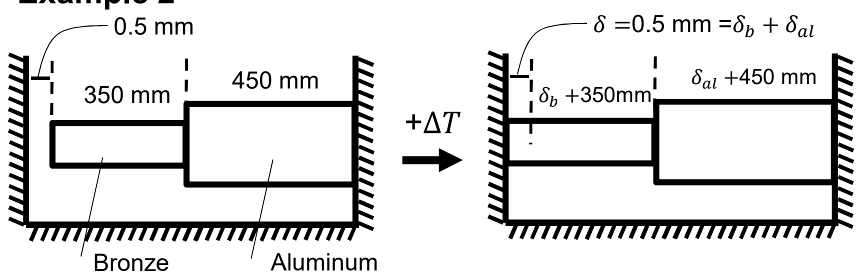
Knowing that a 0.5 mm gap exist between bronze bar and wall when temperature is 24 Celcius, determine

(a) temperature at which the normal stress in Aluminum bar will be -75 MPa
(b) corresponding length of Aluminum bar

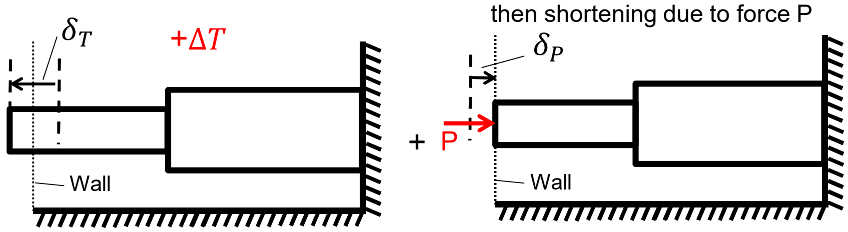
- When **temperature is increased**, the bars will **extend**.
- But they can only extend up to the left wall (Constraint)

- If the bronze bar touches the wall, and temperature keeps increasing then there will be a **reaction force at the wall**
- This reaction force will create normal stress in the bars.

Example 2

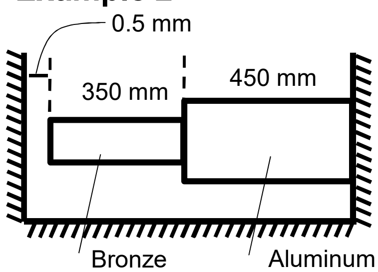


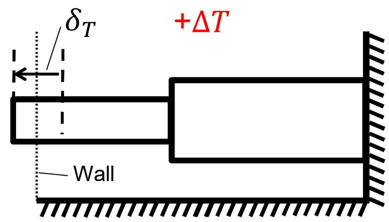
- This can be equivalently represented as follow: Extension due to ΔT + then shortening due to force P



$$\delta = \delta_T - \delta_P$$

Example 2





- Length change due to temperature change can be found with:

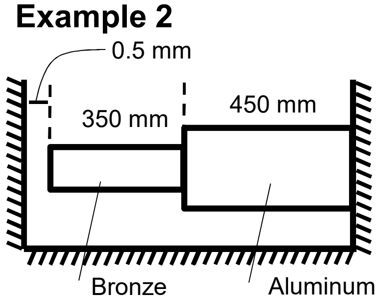
$$\delta_T = L_b \alpha_b \Delta T + L_{al} \alpha_{al} \Delta T$$

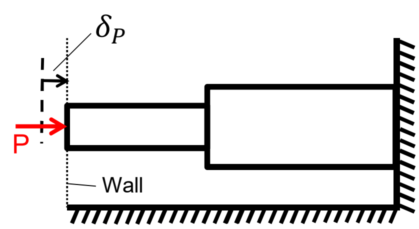
$$\delta_T = (0.35)(21.6x10^{-6})(\Delta T) + (0.45)(23.2x10^{-6})(\Delta T)$$

$$\boldsymbol{\delta_T = (18x10^{-6})(\Delta T)}$$

Example 2

(a) temperature at which the normal stress in Aluminum bar will be -75 MPa





- Length change due to force P can be found with: (from question) $(\sigma = P/A)$

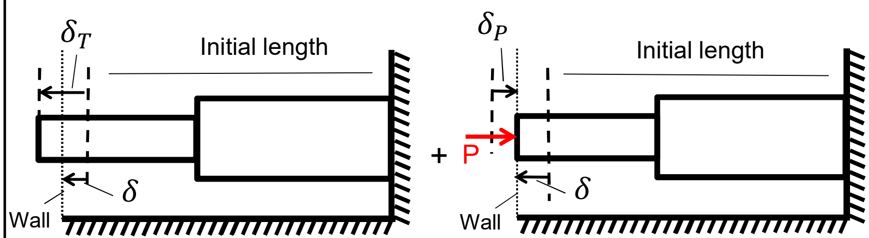
$$\delta_P = \frac{PL_b}{A_b E_b} + \frac{PL_{al}}{A_{al} E_{al}}$$

where $P = \sigma_{al} A_{al} = (-75 \frac{N}{mm^2})(1800 mm^2)$

$P = -135000\ N$

$$\delta_P = \frac{(135000)(0.35)}{(1500x10^{-6})(105x10^9)} + \frac{(135000)(0.45)}{(1800x10^{-6})(73x10^9)} = \mathbf{762.3x10^{-6}\ m = 0.7623\ mm}$$

Example 2



- Compatibility requirement $\delta = \delta_T - \delta_P = 0.5\ mm$

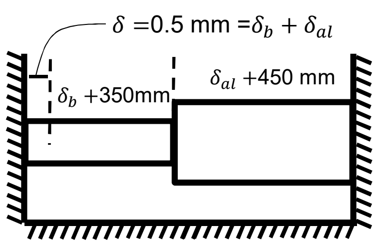
$$\delta = (18x10^{-6}\Delta T) - (762.3x10^{-6}) = 0.5x10^{-3}$$

$$\boldsymbol{\Delta T = 70.1\ C^o}$$

$$\mathrm{T_{new}} = \mathrm{T_0} + \Delta\mathrm{T} = 24 + 70.1 = \mathbf{94.1\ C^o}$$

(a) The temperature at which the normal stress in Aluminum bar will be -75 MPa

Example 2



Knowing that a 0.5 mm gap exist between bronze bar and wall when temperature is 24 Celcius, determine

(a) temperature at which the normal stress in Aluminum bar will be -75 MPa
(b) corresponding length of Aluminum bar

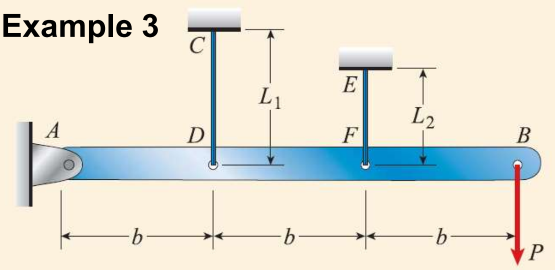
- Now that we know ΔT, we can find δ_b or δ_{al}. (length change of bronze and aluminum bars)
- Similar to part a, $\delta = \delta_T - \delta_P$

$$\delta_{al} = L_{al}\alpha_{al}\Delta T - \frac{PL_{al}}{A_{al}E_{al}} = (0.45)(23.2x10^{-6})(70.1) - \frac{(135000)(0.45)}{(1800x10^{-6})(73x10^9)}$$

$$\delta_{al} = 0.27x10^{-3}\ m$$

$$L_{al,final} = 0.45 + \delta_{al} = 0.45 + 0.27x10^{-3} = \mathbf{0.45027\ m}$$

Example 3

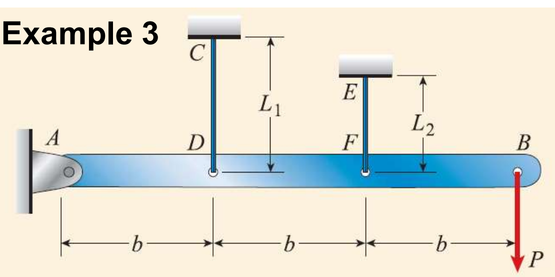


A horizontal rigid bar *AB* is pinned at end *A* and supported by two wires (*CD* and *EF*) at points *D* and *F* (Fig. 2-18a). A vertical load *P* acts at end *B* of the bar. The bar has length $3b$ and wires *CD* and *EF* have lengths L_1 and L_2, respectively. Also, wire *CD* has diameter d_1 and modulus of elasticity E_1; wire *EF* has diameter d_2 and modulus E_2.

(a) Obtain formulas for the allowable load *P* if the allowable stresses in wires *CD* and *EF*, respectively, are σ_1 and σ_2. (Disregard the weight of the bar itself.)

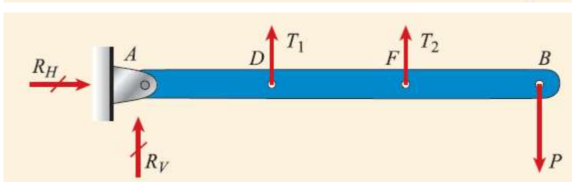
(b) Calculate the allowable load *P* for the following conditions: Wire *CD* is made of aluminum with modulus $E_1 = 72$ GPa, diameter $d_1 = 4.0$ mm, and length $L_1 = 0.40$ m. Wire *EF* is made of magnesium with modulus $E_2 = 45$ GPa, diameter $d_2 = 3.0$ mm, and length $L_2 = 0.30$ m. The allowable stresses in the aluminum and magnesium wires are $\sigma_1 = 200$ MPa and $\sigma_2 = 175$ MPa, respectively.

Example 3



- We need forces in the wires
- Draw free body diagram

Unkowns: $\mathbf{T_1, T_2, R_H, R_V}$



But we can write 3 independent equations of equilibrium:
2 force + 1 moment equilibrium

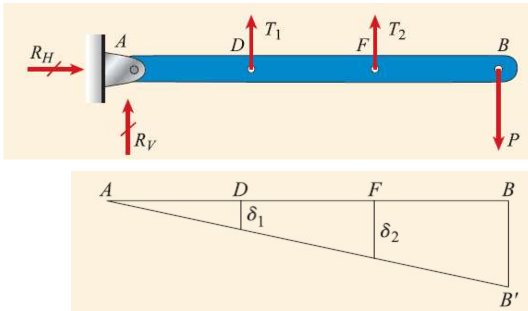
Statically indeterminate
(4 unknowns, 3 equations)

Taking moment w.r.t. A:

$$\sum M_A = 0 \qquad \mathrm{T_1 b + T_2(2b) - P(3b) = 0} \quad \text{or} \quad \mathbf{T_1 + 2T_2 = 3P}$$

- Force equilibrium in horizontal and vertical does not give any usefull information about T_1 *and* T_2.

Example 3

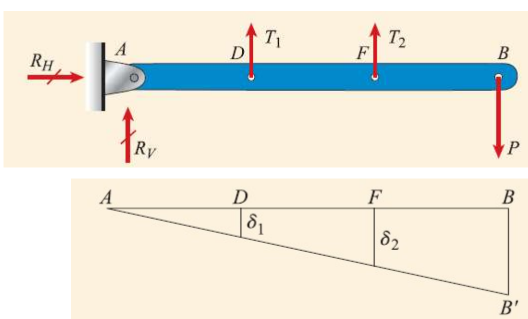


- We can find the additional equation by considering **compatibility**
- **Displacement constraint** because of geometry and using the fact that bar AB is rigid
- Under load P, the bar AB will rotate clockwise, AB' is rotated position

- Since rotation is **very small,** we can assume δ_1, δ_2 are vertical. They are elongations of wires.
- Since |AD|=|DF|, from triangle $2\delta_1 = \delta_2$
- The wires are under axial loading, so:

$$\delta_1 = \frac{T_1 L_1}{A_1 E_1} \quad \delta_2 = \frac{T_2 L_2}{A_2 E_2} \quad \text{where} \quad A_1 = \frac{\pi d_1^2}{4}, A_2 = \frac{\pi d_2^2}{4}$$

Example 3



$$\delta_1 = \frac{T_1 L_1}{A_1 E_1} \quad \delta_2 = \frac{T_2 L_2}{A_2 E_2}$$

- For convenience let:

$$f_1 = \frac{L_1}{A_1 E_1} \quad f_2 = \frac{L_2}{A_2 E_2}$$

$$\delta_1 = T_1 f_1 \quad \delta_2 = T_2 f_2$$

From compatibility: $2\delta_1 = \delta_2 \longrightarrow \mathbf{2T_1 f_1 = T_2 f_2}$

From moment equilibrium: $\mathbf{T_1 + 2T_2 = 3P}$

Solving, we find:

$$T_1 = \frac{3 f_2 P}{4 f_1 + f_2} \quad T_2 = \frac{6 f_1 P}{4 f_1 + f_2}$$

Example 3

(a) Obtain formulas for the allowable load P if the allowable stresses in wires CD and EF, respectively, are σ_1 and σ_2. (Disregard the weight of the bar itself.)

$$T_1 = \frac{3f_2P}{4f_1+f_2} \qquad T_2 = \frac{6f_1P}{4f_1+f_2}$$

We can find stresses in the wires using T_1, T_2

$$\sigma_1 = \frac{T_1}{A_1} = \frac{3P}{A_1}\frac{f_2}{4f_1+f_2} \qquad \sigma_2 = \frac{T_2}{A_2} = \frac{6P}{A_2}\frac{f_1}{4f_1+f_2}$$

σ_1, σ_2 are allowable stresses

- Solve for P

Using σ_1

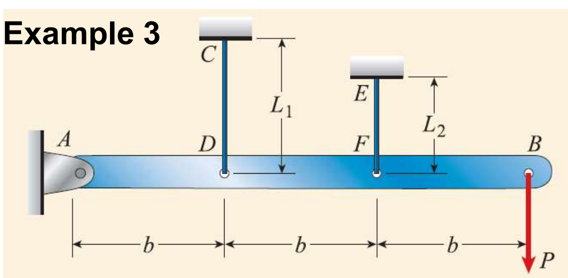
$$P_1 = \frac{\sigma_1 A_1(4f_1+f_2)}{3f_2}$$

Using σ_2

$$P_2 = \frac{\sigma_2 A_2(4f_1+f_2)}{6f_1}$$

- **Smaller of P_1 and P_2 is the allowable load**

Example 3



$E_1 = 72\ GPa \qquad E_2 = 45\ GPa$

$d_1 = 4\ mm \qquad d_2 = 3\ mm$

$L_1 = 0.40\ m \qquad L_2 = 0.30\ m$

$\sigma_1 = 200\ MPa \qquad \sigma_2 = 175\ MPa$

(b) Calculate the allowable load P for the following conditions: Wire CD is made of aluminum with modulus E_1 = 72 GPa, diameter d_1 = 4.0 mm, and length L_1 = 0.40 m. Wire EF is made of magnesium with modulus E_2 = 45 GPa, diameter d_2 = 3.0 mm, and length L_2 = 0.30 m. The allowable stresses in the aluminum and magnesium wires are σ_1 = 200 MPa and σ_2 = 175 MPa, respectively.

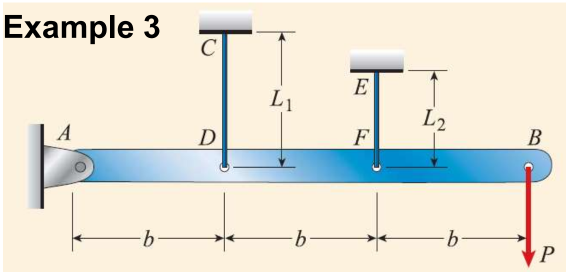
- We have equations;

$$P_1 = \frac{\sigma_1 A_1(4f_1+f_2)}{3f_2}$$

$$P_2 = \frac{\sigma_2 A_2(4f_1+f_2)}{6f_1}$$

$$f_1 = \frac{L_1}{A_1E_1} \qquad f_2 = \frac{L_2}{A_2E_2} \qquad A_1 = \frac{\pi d_1^2}{4}, A_2 = \frac{\pi d_2^2}{4}$$

Example 3



$E_1 = 72\ GPa \qquad E_2 = 45\ GPa$

$d_1 = 4\ mm \qquad d_2 = 3\ mm$

$L_1 = 0.40\ m \qquad L_2 = 0.30\ m$

$\sigma_1 = 200\ MPa \quad \sigma_2 = 175\ MPa$

$$A_1 = \frac{\pi d_1^2}{4} = \frac{\pi(0.004)^2}{4} = 1.2566x10^{-5}$$

$$A_2 = \frac{\pi d_2^2}{4} = \frac{\pi(0.003)^2}{4} = 7.0686\text{x}10^{-6}$$

$$f_1 = \frac{L_1}{A_1 E_1} = \frac{0.4}{(1.2566x10^{-5})(72x10^9)} = 4.4211x10^{-7}$$

$$f_2 = \frac{L_2}{A_2 E_2} = \frac{0.3}{(7.0686x10^{-6})(45x10^9)} = 9.4314x10^{-7}$$

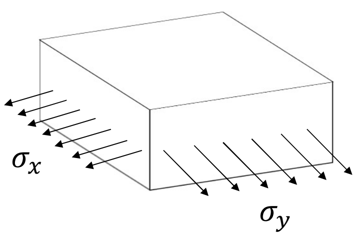
$$P_1 = \frac{\sigma_1 A_1 (4f_1 + f_2)}{3f_2} = \mathbf{2410\ N}$$

$$P_2 = \frac{\sigma_2 A_2 (4f_1 + f_2)}{6f_1} = \mathbf{1260\ N}$$

- Take smaller one

$$\boldsymbol{P_{allow} = 1.26\ kN}$$

Example 4



Assume that ε_x and ε_y are measured experimentally in a test.

$\sigma_x \neq 0$, $\sigma_y \neq 0$ and $\sigma_z = 0$

Show that following relations hold

$$\sigma_x = E\left(\frac{\varepsilon_x + \nu\varepsilon_y}{1 - \nu^2}\right) \qquad \sigma_y = E\left(\frac{\varepsilon_y + \nu\varepsilon_x}{1 - \nu^2}\right)$$

$$\varepsilon_z = -\frac{\nu}{1 - \nu}\left(\varepsilon_x + \varepsilon_y\right)$$

Example 4

Remember multiaxial stress-strain relations

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} \qquad (1)$$

$$\nu\left(\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E}\right) \qquad (2)$$

$$\varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} \qquad (3)$$

Multiply (2) by ν and add to (1)

$$\varepsilon_x + \nu\varepsilon_y = \frac{1-\nu^2}{E}\sigma_x \rightarrow \boxed{\sigma_x = \mathrm{E}\left(\frac{\varepsilon_x + \nu\varepsilon_y}{1-\nu^2}\right)}$$

Multiply (1) by ν and add to (2)

$$\varepsilon_y + \nu\varepsilon_x = \frac{1-\nu^2}{E}\sigma_y \rightarrow \boxed{\sigma_y = \mathrm{E}\left(\frac{\varepsilon_y + \nu\varepsilon_x}{1-\nu^2}\right)}$$

Example 4

$$\boxed{\sigma_x = \mathrm{E}\left(\frac{\varepsilon_x + \nu\varepsilon_y}{1-\nu^2}\right)} \quad \boxed{\sigma_y = \mathrm{E}\left(\frac{\varepsilon_y + \nu\varepsilon_x}{1-\nu^2}\right)}$$

Inserts results into (3)

$$\varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E}$$

$$\varepsilon_z = -\frac{\nu}{E}\left(E\left(\frac{\varepsilon_x + \nu\varepsilon_y}{1-\nu^2}\right) + \mathrm{E}\left(\frac{\varepsilon_y + \nu\varepsilon_x}{1-\nu^2}\right)\right)$$

$$\varepsilon_z = \frac{-\nu\varepsilon_x - \nu^2\varepsilon_y - \nu\varepsilon_y - \nu^2\varepsilon_x}{1-\nu^2}$$

$$\varepsilon_z = \frac{-\nu(\varepsilon_x + \varepsilon_y) - \nu^2(\varepsilon_x + \varepsilon_y)}{1-\nu^2}$$

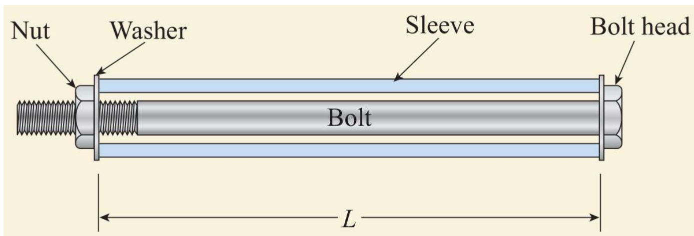
$$\varepsilon_z = -\frac{\nu(\varepsilon_x + \varepsilon_y)(1+\nu)}{(1-\nu)(1+\nu)} \rightarrow \boxed{\varepsilon_z = -\frac{\nu(\varepsilon_x + \varepsilon_y)}{(1-\nu)}}$$

Example 5

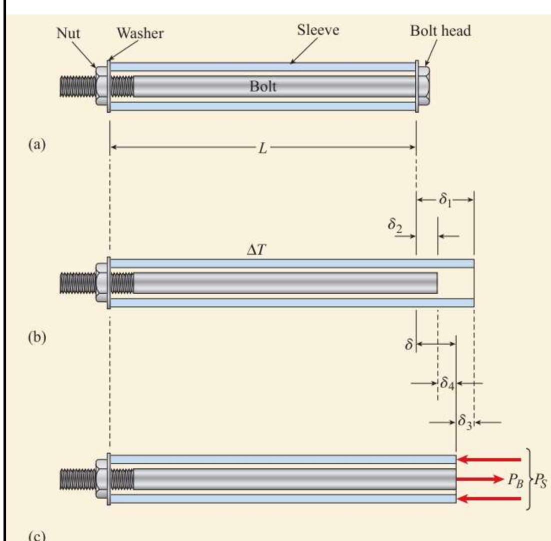
A sleeve in the form of a circular tube of length L is placed around a bolt and fitted between washers at each end (Fig. 2-24a). The nut is then turned until it is just snug. The sleeve and bolt are made of different materials and have different cross-sectional areas. (Assume that the coefficient of thermal expansion α_S of the sleeve is greater than the coefficient α_B of the bolt.)

(a) If the temperature of the entire assembly is raised by an amount ΔT, what stresses σ_S and σ_B are developed in the sleeve and bolt, respectively?

(b) What is the increase δ in the length L of the sleeve and bolt?



Example 5



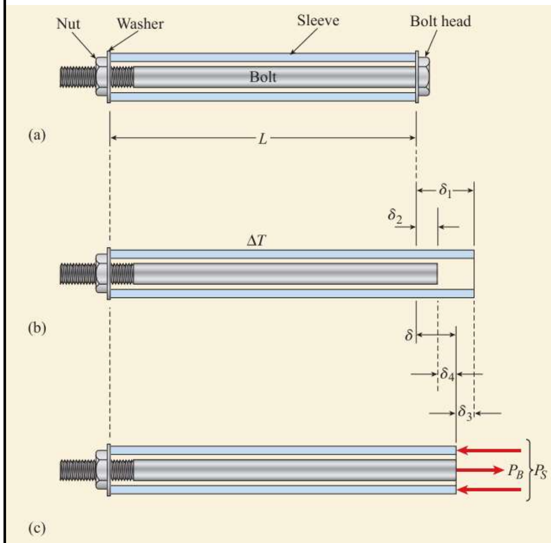
- If free expansion was allowed, the sleeve and bolt would elongate differently.
- However, since they are held together by the assembly, free expansion can not occur and thermal stresses are developed in both materials.

- Temperature-displacement relations are:

$$\delta_1 = \alpha_s(\Delta T)L \qquad \delta_2 = \alpha_B(\Delta T)L$$

Since $\alpha_s > \alpha_B$,
$\delta_1 > \delta_2$

Example 5



- The axial forces in the sleeve and bolt must be such that they shorten the sleeve and bolt and stretch the bolt until final lengths of the sleeve and bolt are the same
- $P_S: Compressive$
  $P_B: Tensile$

$$\delta_3 = \frac{P_S L}{E_S A_S} \qquad \delta_4 = \frac{P_B L}{E_B A_B}$$

- Now we can write an equation of compatibility:

$$\delta = \delta_1 - \delta_3 = \delta_2 + \delta_4$$

Example 5

$$\delta = \delta_1 - \delta_3 = \delta_2 + \delta_4$$

$$\delta = \alpha_S(\Delta T)L - \frac{P_S L}{E_S A_S} = \alpha_B(\Delta T)L + \frac{P_B L}{E_B A_B}$$

$$\frac{P_B L}{E_B A_B} + \frac{P_S L}{E_S A_S} = \alpha_S(\Delta T)L - \alpha_B(\Delta T)L \quad (1)$$

- From equation of equilibrium: $P_S = P_B$ (2)

$$P_B L\left(\frac{1}{E_B A_B} + \frac{1}{E_S A_S}\right) = (\Delta T)L(\alpha_S - \alpha_B)$$

$$\boxed{P_S = P_B = \frac{(\alpha_S - \alpha_B)(\Delta T)E_S A_S E_B A_B}{E_S A_S + E_B A_B}}$$

Example 5

$$\sigma_S = \frac{P_S}{A_S} = \frac{(\alpha_S - \alpha_B)(\Delta T)E_S E_B A_B}{E_S A_S + E_B A_B}$$ Compressive

$$\sigma_B = \frac{P_B}{A_B} = \frac{(\alpha_S - \alpha_B)(\Delta T)E_S A_S E_B}{E_S A_S + E_B A_B}$$ Tensile

Note that:
- These stresses are independent of length
- Their magnitudes are inversely proportional to their respective areas

$$\frac{\sigma_S}{\sigma_B} = \frac{A_B}{A_S}$$

Example 5

- For elongation we can substitute P_B or P_S into δ equation

$$P_S = P_B = \frac{(\alpha_S - \alpha_B)(\Delta T)E_S A_S E_B A_B}{E_S A_S + E_B A_B}$$

$$\delta = \alpha_S(\Delta T)L - \frac{P_S L}{E_S A_S} = \alpha_B(\Delta T)L + \frac{P_B L}{E_B A_B}$$

$$\delta = \alpha_B(\Delta T)L + \frac{(\alpha_S - \alpha_B)(\Delta T)E_S A_S E_B A_B}{E_S A_S + E_B A_B}\frac{L}{E_B A_B}$$

Example 5

$$\delta = \alpha_B(\Delta T)L + \frac{(\alpha_S - \alpha_B)(\Delta T)E_S A_S E_B A_B}{E_S A_S + E_B A_B}\frac{L}{E_B A_B}$$

$$\delta = \frac{\alpha_B(\Delta T)L(E_S A_S + E_B A_B) + (\alpha_S - \alpha_B)(\Delta T)L E_S A_S}{E_S A_S + E_B A_B}$$

$$\boxed{\delta = \frac{(\alpha_S E_S A_S + \alpha_B E_B A_B)(\Delta T)L}{E_S A_S + E_B A_B}}$$