ME201 STATICS

CHAPTER 5
Distributed Forces: Centroids and Centers of Gravity

Contents

Application

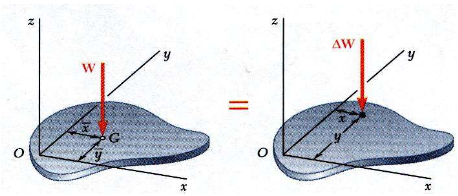
There are many examples in engineering analysis of distributed loads. It is convenient in some cases to represent such loads as a concentrated force located at the *centroid* of the distributed load.

Introduction

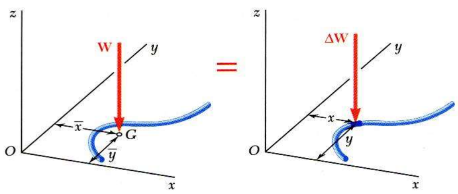
- The **earth exerts a gravitational force on each of the particles forming a body** – consider how your weight is distributed throughout your body. These forces **can be replaced by a single equivalent force equal to the weight of the body and applied at the *center of gravity* for the body.**
- The ***centroid of an area*** is analogous to the **center of gravity of a body**; it is the "center of area." **The concept of the *first moment of an area* is used to locate the centroid.**
- Determination of the **area of a *surface of revolution*** and the **volume of a *body of revolution*** are accomplished with the ***Theorems of Pappus-Guldinus***.

Center of Gravity of a 2D Body

- Center of gravity of a plate



- Center of gravity of a wire



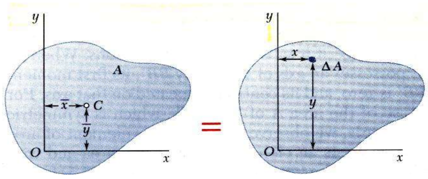
$$\sum M_y \quad \bar{x}W = \sum x\Delta W = \int x\,dW$$

$$\sum M_y \quad \bar{y}W = \sum y\Delta W = \int y\,dW$$

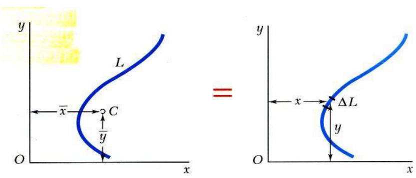
$$\Sigma F_z: \quad W = \Delta W_1 + \Delta W_2 + \cdots + \Delta W_n$$

Centroids and First Moments of Areas and Lines

- Centroid of an area



- Centroid of a line



$$\bar{x}W = \int x\,dW$$

$$\bar{x}(\gamma At) = \int x(\gamma t)dA$$

$$\bar{x}A = \int x\,dA = Q_y$$

$$= \text{first moment with respect to } y$$

$$\bar{y}A = \int y\,dA = Q_x$$

$$= \text{first moment with respect to } x$$

$$\bar{x}W = \int x\,dW$$

$$\bar{x}(\gamma La) = \int x(\gamma a)dL$$

$$\bar{x}L = \int x\,dL$$

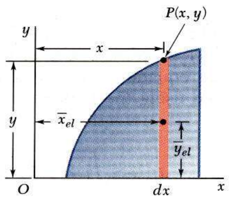
$$\bar{y}L = \int y\,dL$$

γ →Weight per unit volume

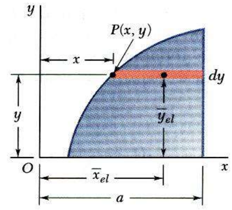
Determination of Centroids by Integration

$$\bar{x}A = \int x\,dA = \iint x\,dx\,dy = \int \bar{x}_{el}\,dA$$
$$\bar{y}A = \int y\,dA = \iint y\,dx\,dy = \int \bar{y}_{el}\,dA$$

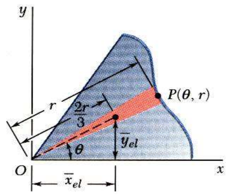
- **Double integration to find the first moment may be avoided by defining *dA* as a thin rectangle or strip.**



$$\bar{x}A = \int \bar{x}_{el}\,dA = \int x\,(y dx)$$
$$\bar{y}A = \int \bar{y}_{el}\,dA = \int \frac{y}{2}(y dx)$$

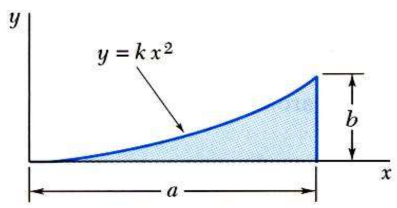


$$\bar{x}A = \int \bar{x}_{el}\,dA = \int \frac{a+x}{2}\left[(a-x)dx\right]$$
$$\bar{y}A = \int \bar{y}_{el}\,dA = \int y\left[(a-x)dx\right]$$



$$\bar{x}A = \int \bar{x}_{el}\,dA = \int \frac{2r}{3}\cos\theta\left(\frac{1}{2}r^2 d\theta\right)$$
$$\bar{y}A = \int \bar{y}_{el}\,dA = \int \frac{2r}{3}\sin\theta\left(\frac{1}{2}r^2 d\theta\right)$$

Sample Problem

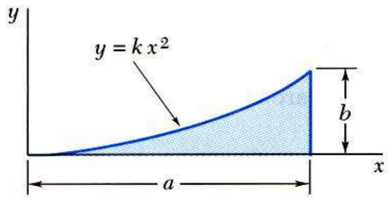


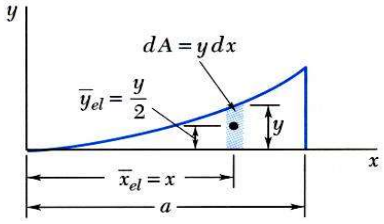
Determine by direct integration the location of the centroid of a parabolic spandrel.

<u>SOLUTION</u>:

- Determine the constant k.
- Evaluate the total area.
- Using either vertical or horizontal strips, perform a single integration to find the first moments.
- Evaluate the centroid coordinates.

Sample Problem





SOLUTION:

- Determine the constant k.

$$y = k\,x^2$$

$$b = k\,a^2 \;\Rightarrow k = \frac{b}{a^2}$$

$$y = \frac{b}{a^2}x^2 \quad or \quad x = \frac{a}{b^{1/2}}y^{1/2}$$

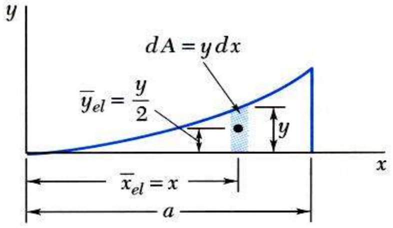
- Evaluate the total area.

$$A = \int dA$$

$$= \int y\,dx = \int_0^a \frac{b}{a^2}x^2 dx = \left[\frac{b}{a^2}\frac{x^3}{3}\right]_0^a$$

$$= \frac{ab}{3}$$

Sample Problem



- Using vertical strips, perform a single integration to find the first moments.

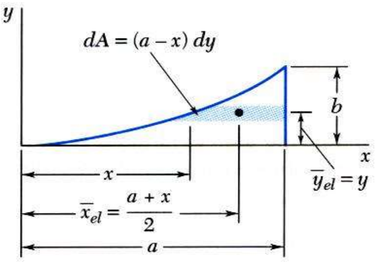
$$Q_y = \int \bar{x}_{el} dA = \int xy\,dx = \int_0^a x\left(\frac{b}{a^2}x^2\right)dx$$

$$= \left[\frac{b}{a^2}\frac{x^4}{4}\right]_0^a = \frac{a^2 b}{4}$$

$$Q_x = \int \bar{y}_{el} dA = \int \frac{y}{2}y\,dx = \int_0^a \frac{1}{2}\left(\frac{b}{a^2}x^2\right)^2 dx$$

$$= \left[\frac{b^2}{2a^4}\frac{x^5}{5}\right]_0^a = \frac{ab^2}{10}$$

Sample Problem



- Or, using horizontal strips, perform a single integration to find the first moments. Try calculating Q_y or Q_x by this method, and confirm that you get the same value as before.

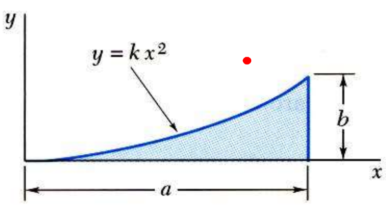
$$Q_y = \int \bar{x}_{el}\, dA = \int \frac{a+x}{2}(a-x)\,dy = \int_0^b \frac{a^2 - x^2}{2}\,dy$$

$$= \frac{1}{2}\int_0^b \left(a^2 - \frac{a^2}{b}y\right)dy = \frac{a^2 b}{4}$$

$$Q_x = \int \bar{y}_{el}\, dA = \int y(a-x)\,dy = \int y\left(a - \frac{a}{b^{1/2}}y^{1/2}\right)dy$$

$$= \int_0^b \left(ay - \frac{a}{b^{1/2}}y^{3/2}\right)dy = \frac{ab^2}{10}$$

Sample Problem



- Evaluate the centroid coordinates.

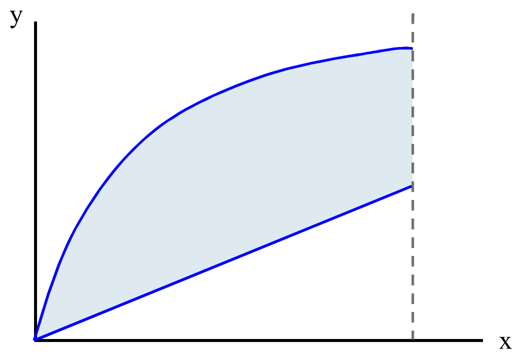
$$\bar{x}A = Q_y$$

$$\bar{x}\frac{ab}{3} = \frac{a^2 b}{4} \qquad \boxed{\bar{x} = \frac{3}{4}a}$$

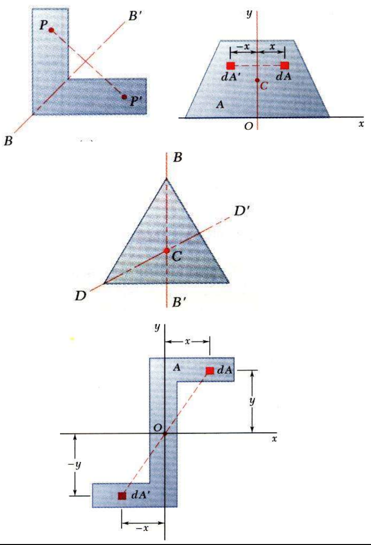
$$\bar{y}A = Q_x$$

$$\bar{y}\frac{ab}{3} = \frac{ab^2}{10} \qquad \boxed{\bar{y} = \frac{3}{10}b}$$

Usually, the choice between using a vertical or horizontal strip is equally good, but in some cases, one choice is much better than the other. For example, for the area shown below, is a vertical or horizontal strip a better choice, and why?

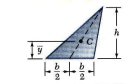
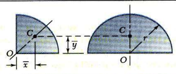
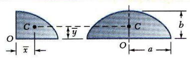
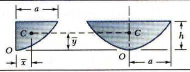
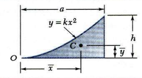
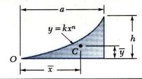
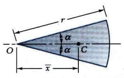


First Moments of Areas and Lines

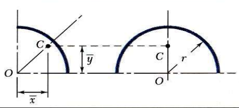
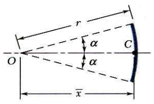


- An area is **symmetric with respect to an axis *BB'*** if **for every point *P* there exists a point *P'* such that *PP'* is perpendicular to *BB'*** and is divided into two equal parts by *BB'*.
- **The first moment of an area with respect to a line of symmetry is zero.**
- **If an area possesses a line of symmetry, its centroid lies on that axis**
- **If an area possesses two lines of symmetry, its centroid lies at their intersection.**
- **An area is symmetric with respect to a center *O* if for every element *dA* at (*x,y*) there exists an area *dA'* of equal area at (*-x,-y*).**
- **<u>The centroid of the area coincides with the center of symmetry.</u>**

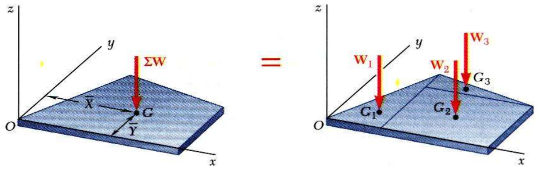
Centroids of Common Shapes of Areas

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r \sin \alpha}{3\alpha}$	0	αr^2

Centroids of Common Shapes of Lines

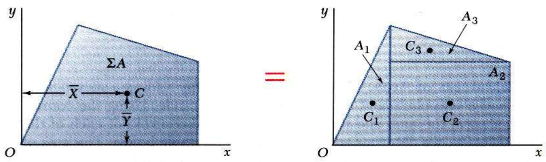
Shape		$\bar{x}$	$\bar{y}$	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

Composite Plates and Areas



- Composite plates

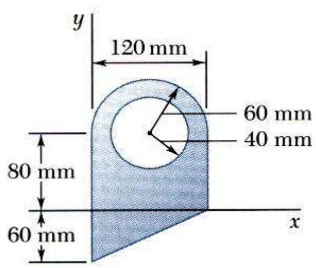
$$\bar{X}\sum W = \sum \bar{x} W$$
$$\bar{Y}\sum W = \sum \bar{y} W$$



- Composite area

$$\bar{X}\sum A = \sum \bar{x} A$$
$$\bar{Y}\sum A = \sum \bar{y} A$$

Sample Problem

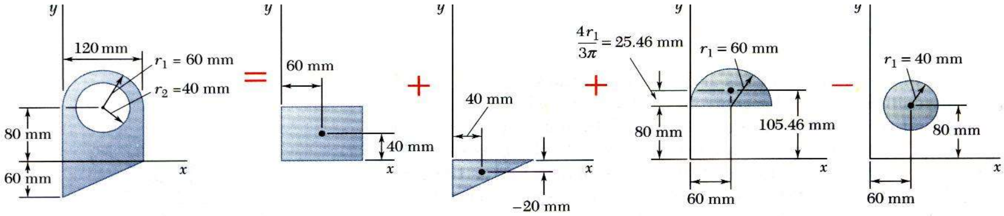


For the plane area shown, determine the first moments with respect to the *x* and *y* axes and the location of the centroid.

SOLUTION:

- Divide the area into a triangle, rectangle, and semicircle with a circular cutout.
- Calculate the first moments of each area with respect to the axes.
- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.
- Compute the coordinates of the area centroid by dividing the first moments by the total area.

Sample Problem



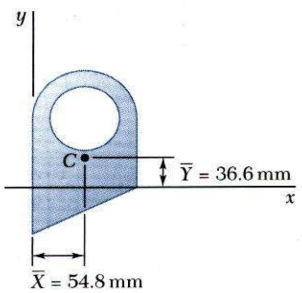
Component	A, mm²	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm³	$\bar{y}A$, mm³
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	−20	$+144 \times 10^3$	-72×10^3
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	-301.6×10^3	-402.2×10^3
	$\Sigma A = 13.828 \times 10^3$			$\Sigma\bar{x}A = +757.7 \times 10^3$	$\Sigma\bar{y}A = +506.2 \times 10^3$

- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.

$$Q_x = +506.2 \times 10^3 \text{ mm}^3$$
$$Q_y = +757.7 \times 10^3 \text{ mm}^3$$

Sample Problem

- Compute the coordinates of the area centroid by dividing the first moments by the total area.



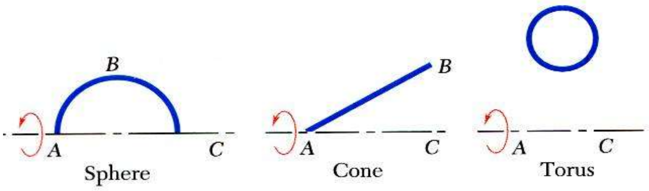
$$\bar{X} = \frac{\sum \bar{x}A}{\sum A} = \frac{+757.7 \times 10^3 \text{ mm}^3}{13.828 \times 10^3 \text{ mm}^2}$$

$$\bar{X} = 54.8 \text{ mm}$$

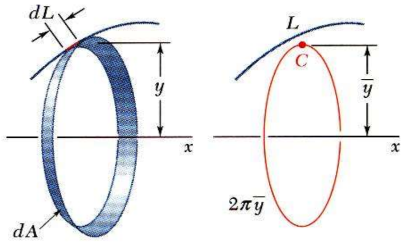
$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{+506.2 \times 10^3 \text{ mm}^3}{13.828 \times 10^3 \text{ mm}^2}$$

$$\bar{Y} = 36.6 \text{ mm}$$

Theorems of Pappus-Guldinus



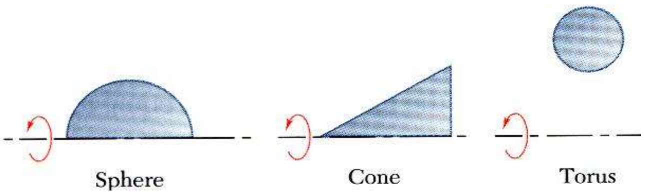
- Surface of revolution is generated by rotating a plane curve about a fixed axis.



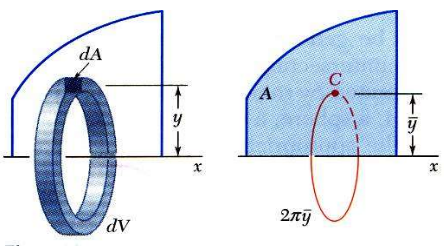
- Area of a surface of revolution is equal to the **length of the generating curve** times the **distance traveled by the centroid through the rotation**.

$$A = 2\pi \bar{y} L$$

Theorems of Pappus-Guldinus



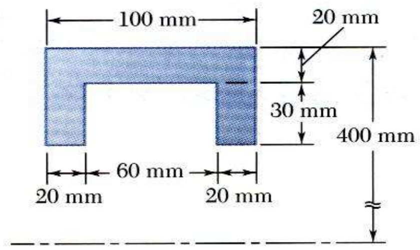
- Body of revolution is generated by rotating a plane area about a fixed axis.



- Volume of a body of revolution is equal to the generating area times the **distance traveled by the centroid through the rotation.**

$$V = 2\pi \bar{y} A$$

Sample Problem



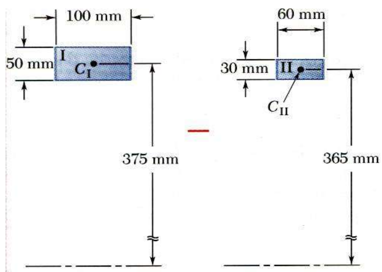
The outside diameter of a pulley is 0.8 m, and the cross section of its rim is as shown. Knowing that the pulley is made of steel and that the density of steel is $\rho = 7.85\times10^3\ \mathrm{kg/m^3}$ determine the mass and weight of the rim.

SOLUTION:

- Apply the theorem of Pappus-Guldinus to evaluate the volumes of revolution of the pulley, which we will form as a large rectangle with an inner rectangular cutout.
- Multiply by density and acceleration to get the mass and weight.

SOLUTION:

- Apply the theorem of **Pappus-Guldinus to evaluate the volumes or revolution for the rectangular rim section and the inner cutout section.**
- Multiply by density and acceleration to get the mass and weight.

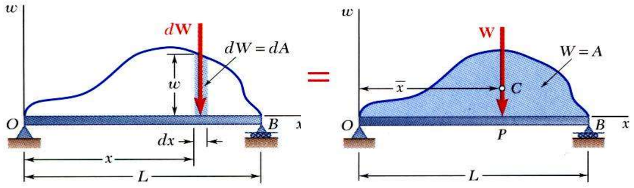


	Area, mm²	$\bar{y}$, mm	Distance Traveled by C, mm	Volume, mm³
I	+5000	375	$2\pi(375) = 2356$	$(5000)(2356) = 11.78\times10^6$
II	−1800	365	$2\pi(365) = 2293$	$(-1800)(2293) = -4.13\times10^6$
				Volume of rim $= 7.65\times10^6$

$$m = \rho V = \left(7.85\times10^3\ \mathrm{kg/m^3}\right)\left(7.65\times10^6\ \mathrm{mm^3}\right)\left(10^{-9}\ \mathrm{m^3/mm^3}\right) \qquad m = 60.0\ \mathrm{kg}$$

$$W = mg = (60.0\ \mathrm{kg})\left(9.81\ \mathrm{m/s^2}\right) \qquad W = 589\ \mathrm{N}$$

Distributed Loads on Beams



$$W = \int_0^L w\,dx = \int dA = A$$

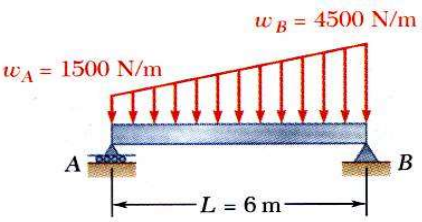
- **A distributed load is represented by plotting the load per unit length, *w* (N/m) . The total load is equal to the area under the load curve.**

$$(OP)W = \int x\,dW$$

$$(OP)A = \int_0^L x\,dA = \bar{x}A$$

- A distributed load can be replace by a **concentrated load with a magnitude equal to the area under the load curve and a line of action passing through the area centroid**.

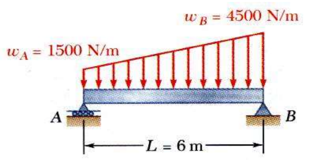
Sample Problem

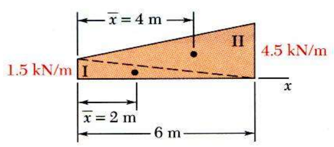


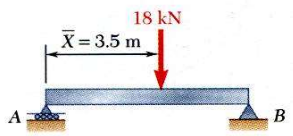
A beam supports a distributed load as shown. Determine the equivalent concentrated load and the reactions at the supports.

SOLUTION:

- The magnitude of the concentrated load is equal to the total load or the area under the curve.
- The line of action of the concentrated load passes through the centroid of the area under the curve.
- Determine the support reactions by (a) drawing the free body diagram for the beam and (b) applying the conditions of equilibrium.







SOLUTION:

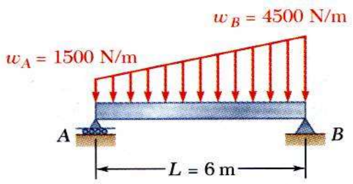
- The magnitude of the concentrated load is equal to the **total load or the area under the curve.**

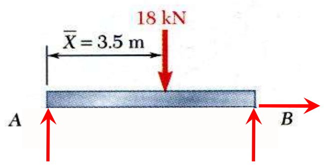
$$F = 18.0\text{ kN}$$

- The line of action of the concentrated load passes through the centroid of the area under the curve.

$$\bar{X} = \frac{63\text{ kN}\cdot\text{m}}{18\text{ kN}} \qquad \bar{X} = 3.5\text{ m}$$

Component	A, kN	$\bar{x}$, m	$\bar{x}A$, kN·m
Triangle I	4.5	2	9
Triangle II	13.5	4	54
	$\Sigma A = 18.0$		$\Sigma\bar{x}A = 63$





- Determine the support reactions by applying the equilibrium conditions. For example, successively sum the moments at the two supports:

$$\sum M_A = 0: \quad B_y(6\text{ m}) - (18\text{ kN})(3.5\text{ m}) = 0$$

$$B_y = 10.5\text{ kN}$$

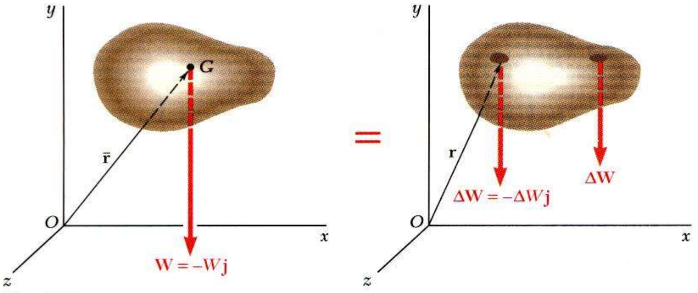
$$\sum M_B = 0: \quad -A_y(6\text{ m}) + (18\text{ kN})(6\text{ m} - 3.5\text{ m}) = 0$$

$$A_y = 7.5\text{ kN}$$

- And by summing forces in the x-direction:

$$\sum F_x = 0: \quad B_x = 0$$

Center of Gravity of a 3D Body: Centroid of a Volume



- Center of gravity G

$$-W\vec{j} = \sum\left(-\Delta W\vec{j}\right)$$

$$\vec{r}_G \times \left(-W\vec{j}\right) = \sum\left[\vec{r}\times\left(-\Delta W\vec{j}\right)\right]$$

$$\vec{r}_G W \times \left(-\vec{j}\right) = \left(\sum \vec{r}\Delta W\right)\times\left(-\vec{j}\right)$$

$$W = \int dW \qquad \vec{r}_G W = \int \vec{r}\,dW$$

- Results are independent of body orientation,

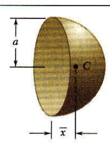
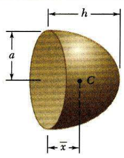
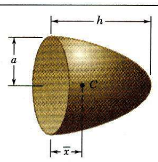
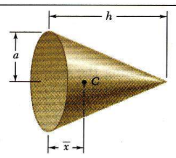
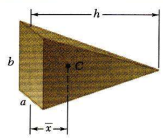
$$\bar{x}W = \int x\,dW \qquad \bar{y}W = \int y\,dW \qquad \bar{z}W = \int z\,dW$$

- For homogeneous bodies,

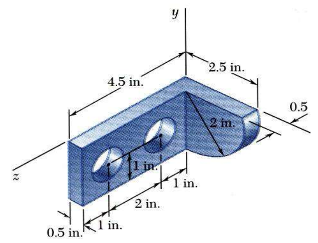
$$W = \gamma V \text{ and } dW = \gamma\,dV$$

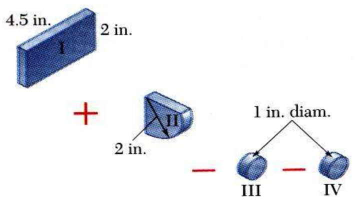
$$\bar{x}V = \int x\,dV \qquad \bar{y}V = \int y\,dV \qquad \bar{z}V = \int z\,dV$$

Centroids of Common 3D Shapes

Shape		$\bar{x}$	Volume
Hemisphere		$\frac{3a}{8}$	$\frac{2}{3}\pi a^3$
Semiellipsoid of revolution		$\frac{3h}{8}$	$\frac{2}{3}\pi a^2 h$
Paraboloid of revolution		$\frac{h}{3}$	$\frac{1}{2}\pi a^2 h$
Cone		$\frac{h}{4}$	$\frac{1}{3}\pi a^2 h$
Pyramid		$\frac{h}{4}$	$\frac{1}{3}abh$

Composite 3D Bodies





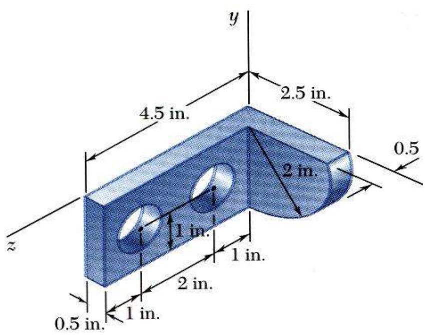
- Moment of the total weight concentrated at the center of gravity G is equal to the sum of the moments of the weights of the component parts.

$$\bar{X}\sum W = \sum \bar{x}W \quad \bar{Y}\sum W = \sum \bar{y}W \quad \bar{Z}\sum W = \sum \bar{z}W$$

- For homogeneous bodies,

$$\bar{X}\sum V = \sum \bar{x}V \quad \bar{Y}\sum V = \sum \bar{y}V \quad \bar{Z}\sum V = \sum \bar{z}V$$

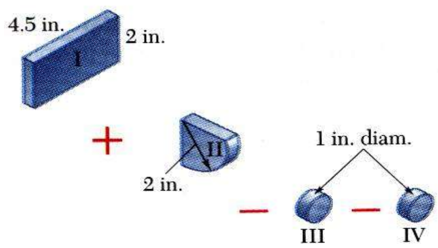
Sample Problem

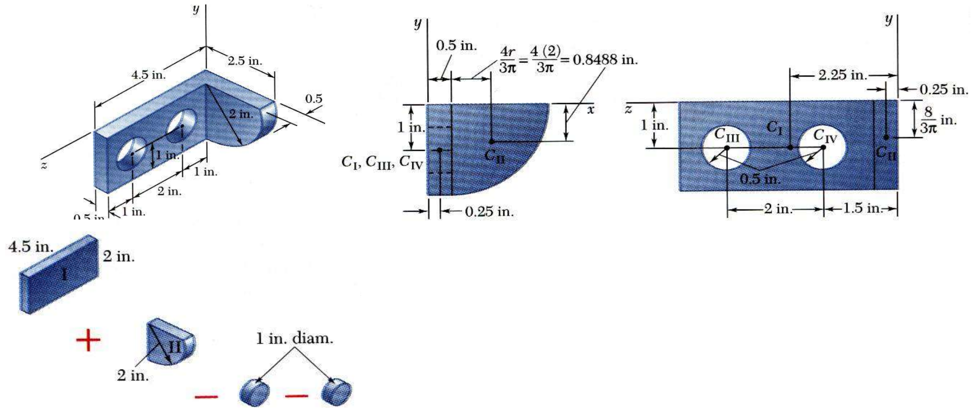


Locate the center of gravity of the steel machine element. The diameter of each hole is 1 in.

SOLUTION:

- Form the machine element from a rectangular parallelepiped and a quarter cylinder and then subtracting two 1-in. diameter cylinders.





	V, in³	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{x}V$, in⁴	$\bar{y}V$, in⁴	$\bar{z}V$, in⁴
I	$(4.5)(2)(0.5) = 4.5$	0.25	−1	2.25	1.125	−4.5	10.125
II	$\frac{1}{4}\pi(2)^2(0.5) = 1.571$	1.3488	−0.8488	0.25	2.119	−1.333	0.393
III	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	−1	3.5	−0.098	0.393	−1.374
IV	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	−1	1.5	−0.098	0.393	−0.589
	$\Sigma V = 5.286$				$\Sigma\bar{x}V = 3.048$	$\Sigma\bar{y}V = -5.047$	$\Sigma\bar{z}V = 8.555$

	V, in³	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{z}$, in.	$\bar{x}V$, in⁴	$\bar{y}V$, in⁴	$\bar{z}V$, in⁴
I	$(4.5)(2)(0.5) = 4.5$	0.25	−1	2.25	1.125	−4.5	10.125
II	$\frac{1}{4}\pi(2)^2(0.5) = 1.571$	1.3488	−0.8488	0.25	2.119	−1.333	0.393
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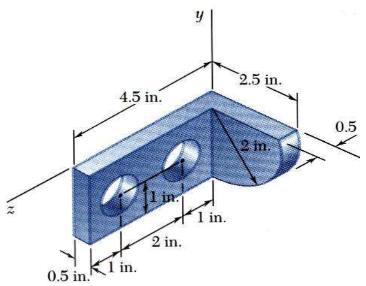
	$\Sigma V = 5.286$				$\Sigma\bar{x}V = 3.048$	$\Sigma\bar{y}V = -5.047$	$\Sigma\bar{z}V = 8.555$



$$\bar{X} = \sum \bar{x}V \Big/ \sum V = \left(3.08 \text{ in}^4\right) \big/ \left(5.286 \text{ in}^3\right)$$

$$\boxed{\bar{X} = 0.577 \text{ in.}}$$

$$\bar{Y} = \sum \bar{y}V \Big/ \sum V = \left(-5.047 \text{ in}^4\right) \big/ \left(5.286 \text{ in}^3\right)$$

$$\boxed{\bar{Y} = 0.577 \text{ in.}}$$

$$\bar{Z} = \sum \bar{z}V \Big/ \sum V = \left(1.618 \text{ in}^4\right) \big/ \left(5.286 \text{ in}^3\right)$$

$$\boxed{\bar{Z} = 0.577 \text{ in.}}$$