CHAPTER 3 | VECTORS

Scalars

- If a single number is sufficient to describe a (physical) quantity, such as the volume of a box or the mass of an object, then that quantity is said to be a **scalar quantity**.

Vectors

- In physics and in all engineering sciences there are a large number of quantities such as position, displacement, velocity, and acceleration that possess both *magnitude* and *direction*. We geometrically represent them by arrows and denote by $\overrightarrow{P_1P_2}$ with the understanding that tail is at the point P_1 and head is at the point P_2, as shown in Figure 1. Such a quantity possessing both magnitude and direction is called a **vector quantity**.

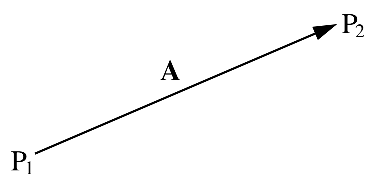

Figure 1: A vector.

- In handwriting we put an arrow over a letter to denote a vector; in print, we use a single letter written in boldface type. For example, the vector in Figure 1 is denoted by either of the notations

$$\overrightarrow{\mathrm{A}} = \overrightarrow{\mathrm{P_1P_2}} \qquad \text{or} \qquad \mathbf{A} = \overrightarrow{\mathrm{P_1P_2}}$$

  We shall almost always use the latter notation, **A**, to represent a vector.

- The *magnitude* of a vector **A** is the length of the arrow (or the size of the quantity which is represented by the vector **A**) and is denoted by one of the following notations:

$$\begin{aligned} A &\equiv |\mathbf{A}| \equiv |\overrightarrow{P_1P_2}| \qquad \text{(in print)} \\ \mathrm{A} &\equiv |\overrightarrow{\mathrm{A}}| \equiv |\overrightarrow{\mathrm{P_1P_2}}| \qquad \text{(in handwriting)} \end{aligned}$$

  We shall use the *slanted* one, A, throughout this notes.

- Though it is superflous to state, it is clear that a vector's magnitude is always a *positive* quantity.

Displacement Vector

- The vector in Figure 1 might represent the **displacement vector** of an object that moves from point $\mathrm{P_1}$ to point $\mathrm{P_2}$, using *any* path between these two points, as shown Figure 2. Therefore, the displacement vector does *not* give any information about the actual path of the object; it tells us only about the *change of position* of the object moving between these two definite points.

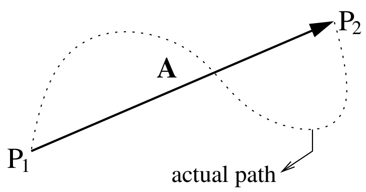

Figure 2: A displacement vector.

- In the following chapter, we shall call the magnitude of the displacement vector as **distance**, and that of the velocity vector as **speed**.

Null Vector

- We define the **null vector**, or **zero vector**, denoted by **0**, as having zero length and no specific direction.

Equal Vectors

- When two vectors **A** and **B** have the same length and the same direction, as shown in Figure 3, they are said to **equal** each other:[1]

$$\mathbf{A} = \mathbf{B}$$

- The exact positions of two equal vectors are immaterial because, except to the position vector defined which will be defined in a moment, *we do not normally ascribe a particular place or position to a vector.* In other words, vectors are *transportable*: you can put a vector wherever you like in a coordinate system, and then you can slide it completely arbitrarily, *parallel to itself*, to another place.

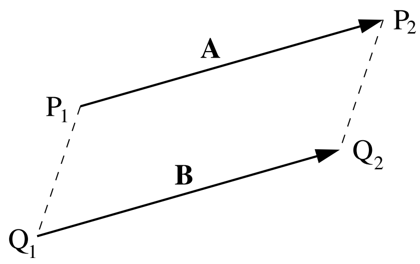

Figure 3: Two equal vectors.

Vector Summation

- The summation (and subtraction) of vectors differ from those of ordinary numbers because these operations require us to take the directions of vectors, their most essential property, into consideration.

[1] An equivalent definition is that two vectors are equal if and only if they have the same components.

- Vectors can be summed together to obtain another vector, which is called their **(vector) sum** or their **resultant**. For two vectors **A** and **B**, we perform this summation geometrically by placing the tail of one vector at the head of the other; their resultant vector starts from the tail of the first vector and terminated at the head of the second vector, as shown in Figure 4.

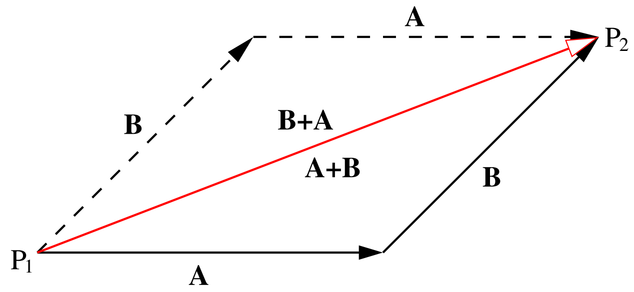

Figure 4: Summation of two vectors. This figure shows also that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

- Keep always in your mind that you can add only vectors of the same kind. You *cannot* add a force vector, for example, to a velocity vector.

- If **A** and **B** are two displacement vectors for an object moving from point P_1 to point P_2 following two different paths, their sum $\mathbf{A} + \mathbf{B}$ gives the *overall* or *net* displacement of the object. Figure 4 exemplifies also such a case.

- We said the summation of two vectors **A** and **B** gives another vector, say **C**. We write mathematically the relation among them as a **vector equation**

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

  Although there appears deceptively a single equation here, it includes actually three distinct equations, one for each dimension x, y, and z, as you will notice clearly later.

- It is an easy task to show that the summation operation is **commutative**; that is, the order of addition is of no importance (Figure 4):

$$\boxed{\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}}$$

- If there are more than two vectors, each vector is summed successively in the tail-to-head fashion just mentioned above; the resultant vector is the one that starts from the tail of the first vector and terminated at the head of the last vector. An example is shown in Figure 5 (a).

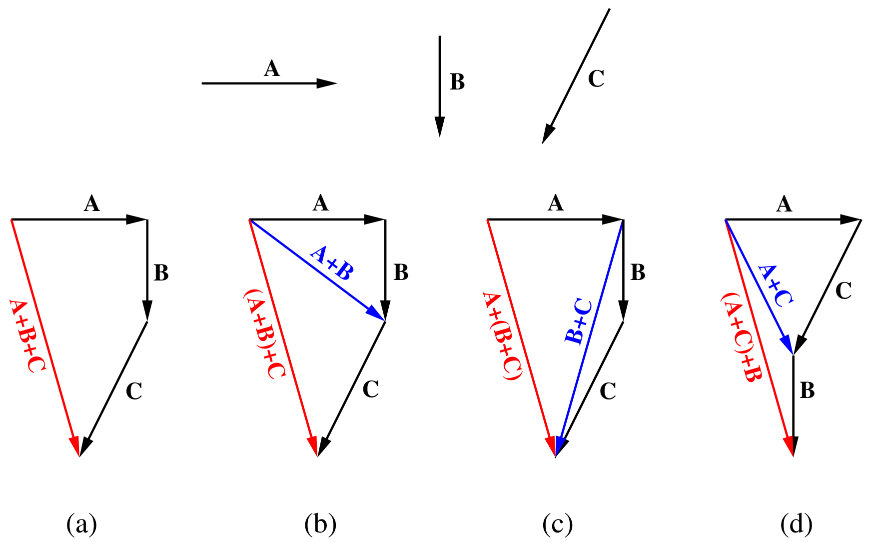

Figure 5: Summation of three vectors.

- We also say that the summation operation is **associative**; that is, if there are three or more vectors to sum, we are free to pick and put any of them into a group and add them in any order we like:

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{C}) + \mathbf{B}$$

The four cases in Figure 5 clearly demostrates this property.

Scalar Multiplication

- By definition, if a vector **A** is multiplied by a *scalar* c, the result is another vector, $c\mathbf{A}$. The magnitude of this new vector is $|c|A$. If $c > 0$, $c\mathbf{A}$ is *parallel to* **A**; if $c < 0$, $c\mathbf{A}$ is *antiparallel to* **A**. Examples of scalar multiplication are given in Figure 6.

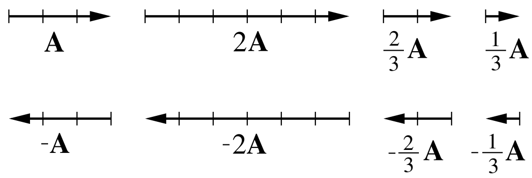

Figure 6: Multiplying a vector with a scalar.

- Having a close look at Figure 6, we can interpret scalar multiplication as *contraction* or *elongation* of a vector.
- It follows from the above definition that multiplying a vector by $c = 0$, the result is the null vector. Multiplication by unity, $c = 1$, leaves the vector unchanged.
- The particular case in which $c = -1$ is especially important, because it leads to the (**additive**) **inverse** of vector $\mathbf{A}$, which is (naturally) denoted by $-\mathbf{A}$.
- It is obvious that the scalar multiplication is **distributive**:

$$c(\mathbf{A} + \mathbf{B}) = (\mathbf{A} + \mathbf{B})c = c\mathbf{A} + c\mathbf{B}$$

Subtraction of Vectors

- Using the notion of scalar multiplication we can **subtract** a vector $\mathbf{B}$ from another vector $\mathbf{A}$ as

$$\boxed{\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})}$$

  Thus, the vector subtraction $\mathbf{A} - \mathbf{B}$ can be seen as the vector summation of $\mathbf{A}$ and the inverse vector $-\mathbf{B}$.
- Figure 7 shows how this is done geometrically. An easy way of remembering the subtraction $\mathbf{A}_1 - \mathbf{A}_2$ is by placing the two vectors tail–to–tail. The desired difference vector is the one from the second vector $\mathbf{A}_2$ to the first one $\mathbf{A}_1$. The last two depictions in Figure 7 illustrate this easy way.

Resolving a Vector Lying on an xy-Plane

- If we **resolve** (or **decompose**) a vector into its **components**, we find its projections on the axes of a coordinate system, which are found by drawing a perpendicular line from the head of the vector to the axes.
- Consider a vector $\mathbf{A}$ lying on an xy-plane, as in Figure 8 (a). Its projection on the x-axis is referred to as its *x-component*, $\mathbf{A}_x$, and that on the y-axis as its *y-component*, $\mathbf{A}_y$.

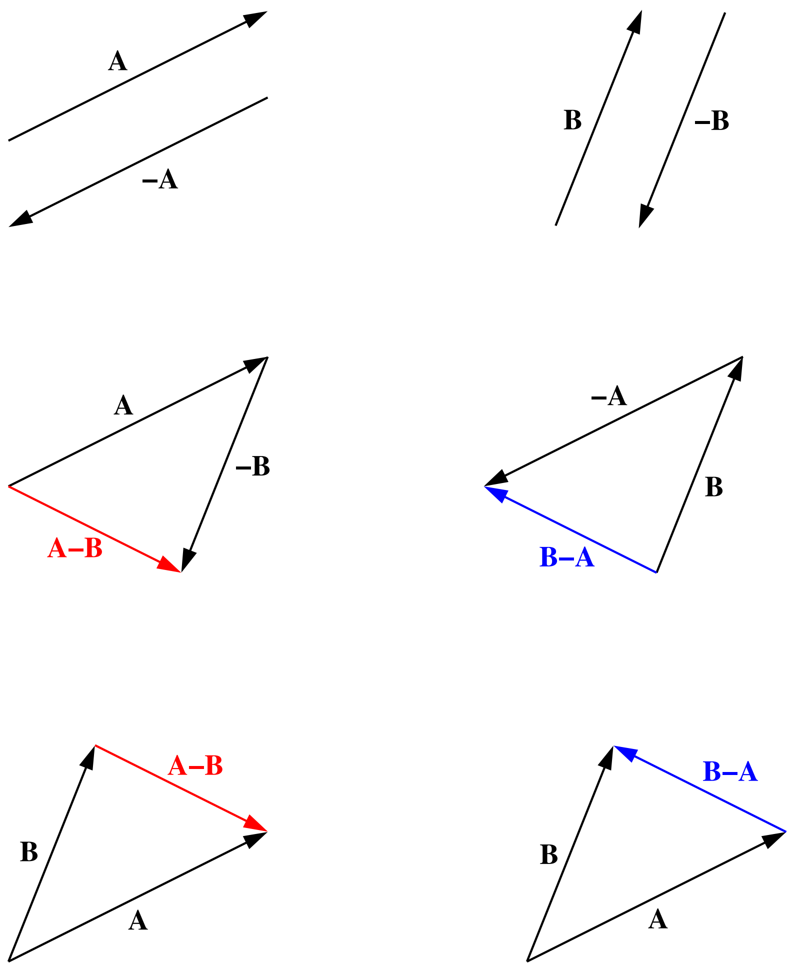

Figure 7: The subtractions $\mathbf{A} - \mathbf{B}$ and $\mathbf{B} - \mathbf{A}$ for two vectors $\mathbf{A}$ and $\mathbf{B}$.

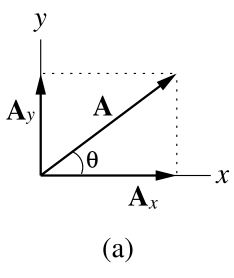

Figure 8: Components of a vector in an xy-plane.

- Since a vector is transportable, you can slide the component $\mathbf{A}_y$ in Figure 8 (a) to the right to obtain the right triangle in Figure 8 (b). It follows from this figure that vector $\mathbf{A}$ equals the vector sum of components $\mathbf{A}_x$ and $\mathbf{A}_y$:

$$\boxed{\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y}$$

- The magnitudes of the components of $\mathbf{A}$ are seen to be[2]

$$\boxed{A_x = A\cos\theta \quad \text{and} \quad A_y = A\sin\theta}$$

The relation between the three magnitudes in Figure 8 (b) is found by the Pythagorean theorem as

$$\boxed{A = \sqrt{A_x^2 + A_y^2}}$$

and angle θ is determined from[3]

$$\boxed{\tan\theta = \frac{A_y}{A_x} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)}$$

- It might be useful to notice for your future advanced courses that a vector $\mathbf{A}$ in an xy-plane can be *completely specified* if we know (1) its components A_x and A_y or (2) its magnitude A and the angle θ it makes with positive x-axis. Pairs (A_x, A_y) and (A, θ) both convey the same information about vector $\mathbf{A}$. We refer to the former, (A_x, A_y), as the **component notation** whereas the latter, (A, θ), as the **magnitude-angle notation**.

[2]It is very important to note that these formulas are valid if and only if angle θ is measured from the positive x-axis. If this angle were measured from the positive y-axis, these formulas would appear as $A_x = A\sin\theta$ and $A_y = A\cos\theta$. And again the moral is that don't memorize any formula, learn them by heart!

[3]It is a convention to measure an angle, in an xy-plane, with respect to the positive x-axis. This angle is taken to be positive if it is measured in counterclockwise sense and negative if measured clokwise. With these conventions, $290°$ and $-70°$, for example, are the same angle.

- Summation of vectors can be carried out pictorially in a simpler and more comprehensible way, as is done in Figure 9. Suppose we are after the result of the summation

$$\mathbf{R} = \mathbf{A} + \mathbf{B}$$

As is clearly seen from Figure 9, the x- and y-components of the resultant vector $\mathbf{R}$ are

$$R_x = A_x + B_x \quad \text{and} \quad R_y = A_y + B_y$$

and its magnitude is

$$R = \sqrt{R_x^2 + R_y^2}$$

and finally the angle that $\mathbf{R}$ makes with the positive x-axis is found from

$$\tan\theta = \frac{R_y}{R_x} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

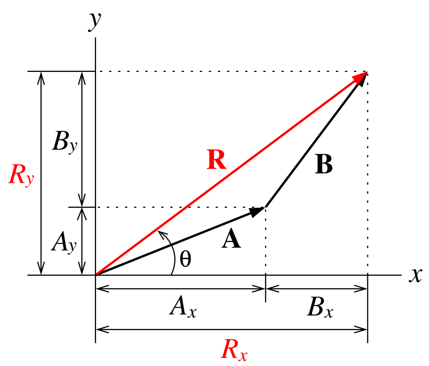

Figure 9: A vector its components on an xy-plane.

Unit Vectors

- The **unit vector** $\hat{\mathbf{A}}$ of the *nonzero* vector $\mathbf{A}$ is defined to have a length exactly 1 and to be in the same direction as $\mathbf{A}$. We form $\hat{\mathbf{A}}$ by dividing $\mathbf{A}$ by its length A:

$$\boxed{\hat{\mathbf{A}} = \frac{\mathbf{A}}{A}}$$

- As an attentive student may easily notices, a unit vector does *not* have a dimension nor a unit.

- A unit vector does nothing but specifies a direction. But this fact does not lessen at all the importance of unit vectors. Let me rewrite the above equation as

  $$\boxed{\mathbf{A} = A\hat{\mathbf{A}}}$$

  I can readily say that this is the most important equation about vectors because the two most essential properties of a vector are embodied in this equation: its magnitude and its direction. Put in another way, a vector is the product of its magnitude and its direction. Now the vector in Figure 10 should appear more meaningful.

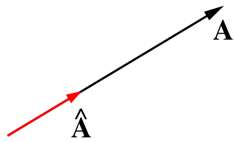

Figure 10: A vector with its unit vector pointing its direction.

- In the usual **right-handed rectangular** xyz**-coordinate system**,[4] we shall use three special **cartesian unit vectors i**, **j**, and **k** that point the positive directions of the x-, y, and z-axes, respectively.[5] More specifically, the vector **i** is from origin to the point (1,0,0), **j** is from origin to the point (0,1,0), and **k** is from origin to the point (0,0,1), as they are drawn in Figure 11. Then we write any vector **A** as a linear combination of these **basis vectors** as

  $$\boxed{\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}}$$

  A vector given in this form is said to be written **in unit-vector notation**.

[4] To check whether this coordinate system is indeed right-handed, we write the names of the axes in sequence like $\ldots xyzxyz\ldots$. Then we pick any three of them in this order, say z, x, and y. Now if we sweep the z-axis with our right hand towards the x-axis, the upright thumb should point in the same direction as the y, and this is indeed the case so that we have a right-handed coordinate system. You should demonstrate that you *cannot* obtain the same result with our left hand! Note that the choice of the axes' names and their order have not been made haphazardly at all.

[5] In handwriting these unit vectors appear as $\hat{\mathrm{i}}$, $\hat{\mathrm{j}}$, and $\hat{\mathrm{k}}$.

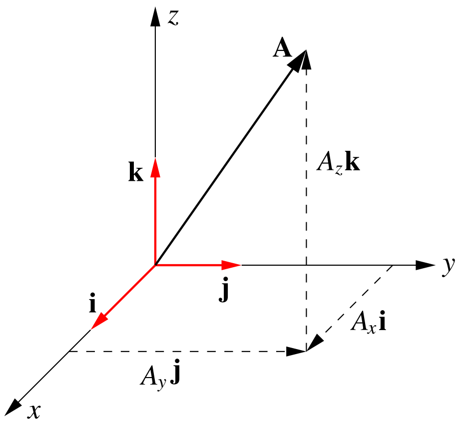

Figure 11: A vector and its components in terms of unit vectors.

- As in Figure 12, for a 2D vector lying on the xy-plane which you will often work with, the above relation reduces to $\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j}$. Recalling $A_x = A\cos\theta$ and $A_y = A\sin\theta$, we obtain the following useful relations to learn by heart:

$$\begin{aligned}\mathbf{A} &= A_x\,\mathbf{i} + A_y\,\mathbf{j}\\ &= A\cos\theta\,\mathbf{i} + A\sin\theta\,\mathbf{j}\\ &= A\,(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})\end{aligned}$$

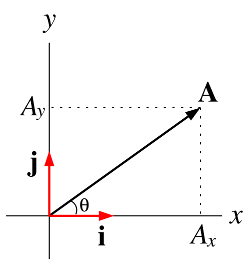

Figure 12: A vector its components on an xy-plane.

- Since a vector is the product of its magnitude and its direction, i.e., $\mathbf{A} = A\hat{\mathbf{A}}$, the last relation above provides us with an equation for the unit vector $\hat{\mathbf{A}}$:[6]

$$\mathbf{A} = A\hat{\mathbf{A}} \quad \text{where} \quad \hat{\mathbf{A}} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

[6]Note once more again that in these and in the preceding ones the formulas containing θ are valid if and only if angle θ is measured from the positive x-axis.

- Let me return to Figure 9 where we had considered pictorially the vector sum, in an xy-plane, $\mathbf{R} = \mathbf{A} + \mathbf{B}$ in terms of the components of the individual vectors. Now we perform this addition with the help of unit-vector notation. We have

$$R_x = A_x + B_x \quad \text{and} \quad R_y = A_y + B_y$$

so that

$$\mathbf{R} = R_x\,\mathbf{i} + R_y\,\mathbf{j} = (A_x + B_x)\,\mathbf{i} + (A_y + B_y)\,\mathbf{j}$$

The magnitude of the resultant vector $\mathbf{R}$ and the angle that $\mathbf{R}$ makes with the positive x-axis is written as before:

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}$$

and

$$\tan\theta = \frac{R_y}{R_x} = \frac{A_y + B_y}{A_x + B_x}$$

or

$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) = \tan^{-1}\left(\frac{A_y + B_y}{A_x + B_x}\right)$$

Vectors in Space: 3D Generalization

- So far we have given the essential properties vectors, mainly in terms of plane vectors lying on an xy-plane. In these sections we shall generalize our preceding results to 3D space vectors with the help of cartesian unit vectors **i**, **j**, and **k**.

- For the sake of completeness, we begin by repeating the relation giving a vector **A** as a linear combination of these unit vectors:

$$\boxed{\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}}$$

- Then in terms of its components, the magnitude (or length) of **A** is seen to be

$$\boxed{A = \sqrt{A_x^2 + A_y^2 + A_z^2}}$$

- For the **null** (**zero**) **vector**, **0**, we now have

$$\mathbf{0} = 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k}$$

- The **summation** and **subtraction** of two vectors $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$ is given as

$$\begin{aligned} \mathbf{R} &= \mathbf{A} \pm \mathbf{B} \\ &= (A_x \pm B_x)\,\mathbf{i} + (A_y \pm B_y)\,\mathbf{j} + (A_z \pm B_z)\,\mathbf{k} \\ &= R_x\,\mathbf{i} + R_y\,\mathbf{j} + R_y\,\mathbf{k} \end{aligned}$$

where the magnitude of the resultant vector $\mathbf{R}$ is

$$\begin{aligned} R &= \sqrt{R_x^2 + R_y^2 + R_z^2} \\ &= \sqrt{(A_x \pm B_x)^2 + (A_y \pm B_y)^2 + (A_z \pm B_z)^2} \end{aligned}$$

- For a given vector $\mathbf{A}$ and a **scalar** c, a real number, we now define **scalar multiplication** as

$$\begin{aligned} c\mathbf{A} &= cA_x\,\mathbf{i} + cA_y\,\mathbf{j} + cA_z\,\mathbf{k} \\ |c\mathbf{A}| &= |c|A \end{aligned}$$

It is obvious that the scalar multiplication is **distributive**:

$$c(\mathbf{A} + \mathbf{B}) = (\mathbf{A} + \mathbf{B})c = c\mathbf{A} + c\mathbf{B}$$

Position Vector

- Although do not normally associate a vector with a definite position, there is a particular example, called the **position vector**, whose tail is at the origin and whose head is at the point $P(x, y, z)$ (See Figure 13). The position vector is commonly represented by $\mathbf{r}$:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

- The position vector $\mathbf{r}$ is in general used to denote the position of a moving object at any time, as we shall see in the next chapter.

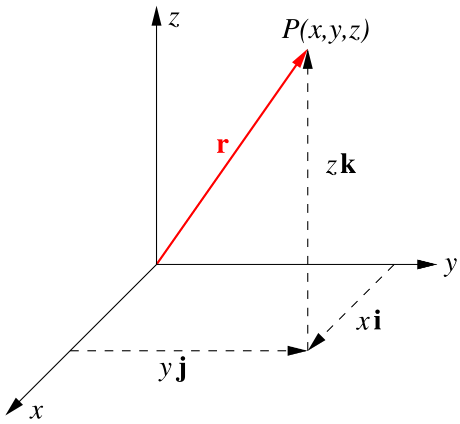

Figure 13: The position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Displacement Vector: A Revisiting

- Consider an object that starts to move from point P_1 and arrives at point P_2, as shown in Figure 14. The so-called **displacement vector**, denoted by $\Delta\mathbf{r}$, which we have mentioned in the beginning of this chapter, gives the net change in the position of the object.

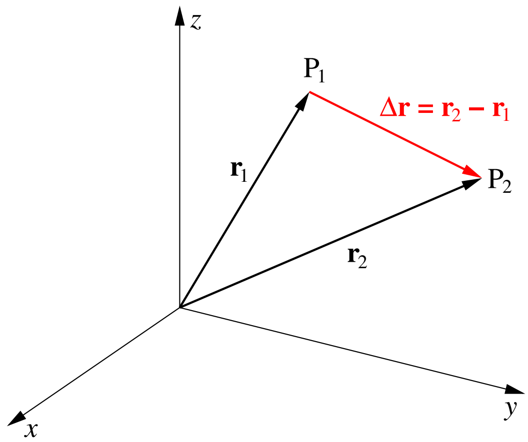

Figure 14: The displacement vector $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$.

- It follows from Figure 14 that $\Delta\mathbf{r}$ amounts to the vector $\mathbf{r}_2 - \mathbf{r}_1$. Therefore, making use of the notion of vector subtraction, we can express $\Delta\mathbf{r}$ as

$$\begin{aligned}\Delta\mathbf{r} &= \mathbf{r}_2 - \mathbf{r}_1 \\ &= (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k}\end{aligned}$$

- Sometimes it is much more meaningful to learn the displacement vector $\Delta\mathbf{r}$ as

$$\mathbf{r}_1 + \Delta\mathbf{r} = \mathbf{r}_2$$

  This relation has the plain interpretation that if an object initially at position $\mathbf{r}_1$ undergoes a displacement $\Delta\mathbf{r}$, it arrives at position $\mathbf{r}_2$.

The Scalar Product of Two Vectors

- Given two vectors $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$, we define their **scalar product** $\mathbf{A}\cdot\mathbf{B}$, to be read as "A dot B", as the sum of the products of their corresponding components:[7]

$$\mathbf{A}\cdot\mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

  As the name suggests, the result is just a *scalar*, a real number.

- Due to the symbol "·" in its notation, the scalar product is also referred to as the **dot product**. The other common name, especially in our country, is the **inner product**.

[7] There are only three types of multiplication defined on vectors; they are scalar multiplication, $c\mathbf{A}$, scalar product, $\mathbf{A}\cdot\mathbf{B}$, and cross product, $\mathbf{A}\times\mathbf{B}$, that you will see shortly. Thus, if you happen to be given two vectors $\mathbf{A}$ and $\mathbf{B}$ and required to evaluate the "product" $\mathbf{AB}$, you should immediately reject this demand with a stern look!

- A comparison of the equation $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$ and the above equation reveals that the magnitude of vector $\mathbf{A}$ is the square root of the scalar product $\mathbf{A} \cdot \mathbf{A}$:

$$\boxed{A = \sqrt{\mathbf{A} \cdot \mathbf{A}} \qquad \text{or} \qquad A^2 = \mathbf{A} \cdot \mathbf{A}}$$

- It follows from the definition that we have for the standard basis vectors that

$$\boxed{\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1}$$

  and

$$\boxed{\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = \mathbf{i} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{j} = 0}$$

- From its definition on the previous page, the scalar product is seen to be commutative and distributive:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= \mathbf{B} \cdot \mathbf{A} \\ \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) &= (\mathbf{B} + \mathbf{C}) \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \\ c\,(\mathbf{A} \cdot \mathbf{B}) &= (\mathbf{A} \cdot \mathbf{B})\,c = (c\,\mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (c\,\mathbf{B}) \end{aligned}$$

- Making use of the Cosine law[8] we can show that

$$\boxed{\mathbf{A} \cdot \mathbf{B} = AB \cos\theta}$$

  where θ is the angle between the *positive* directions of $\mathbf{A}$ and $\mathbf{B}$. For the verification of this relation, have a look at Figure 16. We immediately

[8] For the triangle in Figure 15 the following formulas can be shown to hold:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\,\cos A \\ b^2 &= a^2 + c^2 - 2ac\,\cos B \\ c^2 &= a^2 + b^2 - 2ab\,\cos C \end{aligned}$$

have

$$\begin{aligned} A^2 + B^2 - 2AB\cos\theta &= |\mathbf{A} - \mathbf{B}|^2 \\ &= (\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) \\ &= \mathbf{A} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{B} - \mathbf{B} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B} \\ &= A^2 - 2\mathbf{A} \cdot \mathbf{B} - B^2 \\ \Rightarrow \quad -2AB\cos\theta &= -2\mathbf{A} \cdot \mathbf{B} \\ AB\cos\theta &= \mathbf{A} \cdot \mathbf{B} \qquad \surd \end{aligned}$$

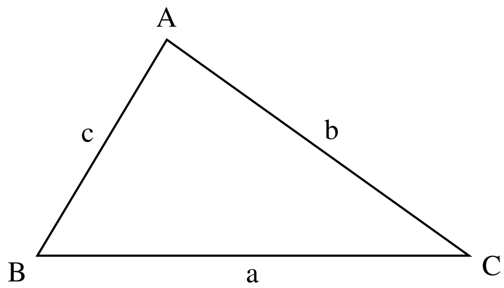

Figure 15: The triangle for the cosine law.

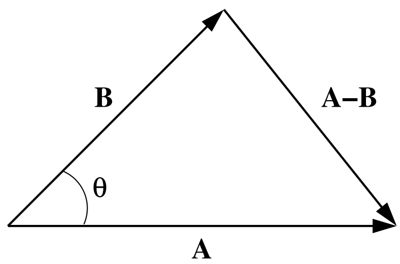

Figure 16: The triangle for the derivation of scalar product.

- Another interpretation of the scalar product comes from Figure 17: we see that $A\cos\theta$ is the projection of **B** on **A**; similarly, $B\cos\theta$ is the projection of **B** on **A**. Consequently, the scalar product $\mathbf{A} \cdot \mathbf{B}$ is numerically amounts to the length of **B** times the projection of **A** on **B**, and also amounts to the length of **A** times the projection of **B** on **A**. If $\theta < 90°$, the projections are *positive*; and if $\theta > 90°$, they are *negative*.

- For the particular case in which the two vectors are *perpendicular*, i.e., $\theta = 90°$, their scalar products is *zero*. This is also obvious from Figure 17.

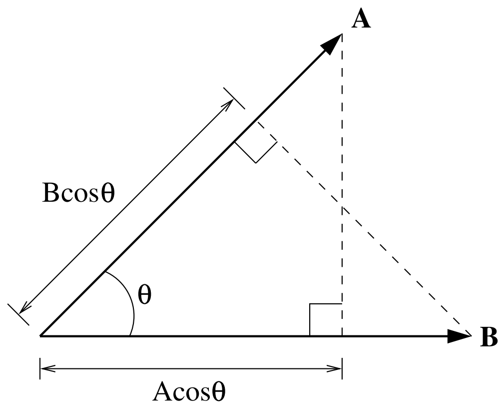

Figure 17: An interpretation of scalar product.

- If two vectors **A** and **B** *parallel*, i.e., $\theta = 0$, their scalar product is AB, if they are antiparallel, i.e., $\theta = 180°$, the corresponding scalar product is equal to $-AB$. Thus, the conclusion is that their scalar products vary between $-AB$ and AB:

$$-AB \leq \mathbf{A} \cdot \mathbf{B} \leq AB$$

The Vector Product of Two Vectors

- Another type of product of two vectors is the **vector product** which, by definition, satisfies the following three conditions:

  1. The vector product of two vectors **A** and **B**, written $\mathbf{A} \times \mathbf{B}$ results in the unique *vector* whose magnitude is

  $$|\mathbf{A} \times \mathbf{B}| = AB \sin\theta$$

  where θ is the *smaller* of the two angles between **A** and **B**, see Figure 18.

  The two other common names for the vector product are the **cross product** (because of the symbol "×" in its notation) and the **outer product**; the latter is heard especially in our country.

  2. The vector $\mathbf{A} \times \mathbf{B}$ is perpendicular to the plane spanned by **A** and **B**; this means that

  $$\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{0} \quad (1)$$

  where **0** is the *null vector*.

3. The direction of $\mathbf{A} \times \mathbf{B}$ is determined according to the **right–hand rule**: Slide **A** and **B**, parallel to themselves, so that they touch tail–to–tail. Then, draw an imaginary perpendicular line to the point where their tails meet. Now imagine that you grasp this line with your *right* hand and sweep your fingers from **A** to **B** through the *smaller* angle between them. Your upright thumb then shows the direction of $\mathbf{A} \times \mathbf{B}$. (See Figure 18.) In other words, **A**, **B** and $\mathbf{A} \times \mathbf{B}$ form a **right–handed triad**.

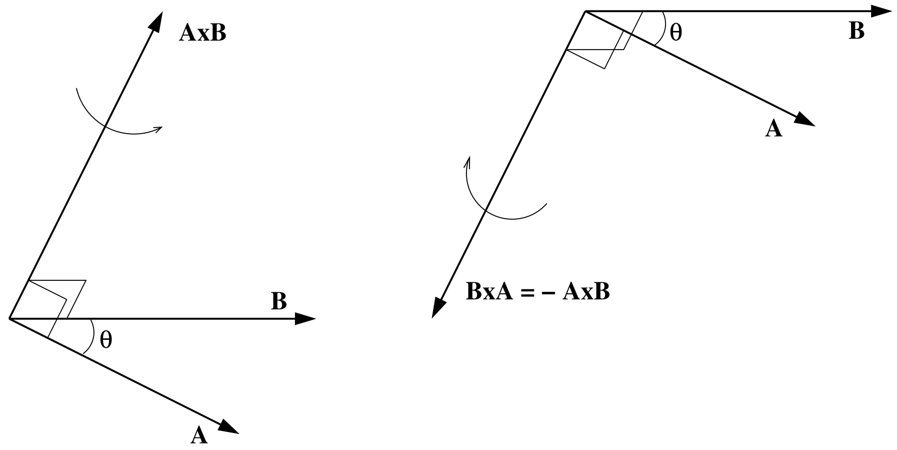

Figure 18: The vector products $\mathbf{A} \times \mathbf{B}$ and $\mathbf{B} \times \mathbf{A}$.

- For the direction of $\mathbf{A} \times \mathbf{B}$, you can also imagine a right-handed screw: the vector $\mathbf{A} \times \mathbf{B}$ shows the direction toward which the screw would advance if **A** were rotated toward **B** through the *smaller* angle between them.

- It directly follows from the equation

$$|\mathbf{A} \times \mathbf{B}| = AB \sin\theta$$

that if two vectors **A** and **B** are parallel, their vector product is the *null vector*:

$$\boxed{\mathbf{A} \times \mathbf{B} = \mathbf{0} \quad \text{if} \quad \mathbf{A} \parallel \mathbf{B}}$$

For the case in which the two vectors are perpendicular, we have for the *magnitude* of their product that

$$\boxed{|\mathbf{A} \times \mathbf{B}| = AB \quad \text{if} \quad \mathbf{A} \perp \mathbf{B}}$$

- The first important property of the vector product is that it is **anti-commutative**:

$$\boxed{\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}}$$

which is seen from the geometric consideration, see Figure 18.[9] This means that you must pay attention to the order of vectors in a vector product.

- Furthermore, the vector product is, in general, *not* associative with respect to the vector product:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$$

(This type of vector products is known as the **triple cross product** and is occasionally important in many areas of physics and engineering sciences. We shall not say more about this type of product since it is out of skope of our introductory physics courses.)

- However, the vector product *is* associative with respect to the multiplication of a constant c:

$$c\,(\mathbf{A} \times \mathbf{B}) = (c\,\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (c\,\mathbf{B})$$

- Besides, the vector product is **distributive** also:

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \\ (\mathbf{A} + \mathbf{B}) \times \mathbf{C} &= \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C} \end{aligned}$$

This can be verified by geometrical considerations. Later you will be able to prove these properties by algebraic manipulations.

- From the definition

$$|\mathbf{A} \times \mathbf{B}| = AB \sin\theta$$

we see that the magnitude $|\mathbf{A} \times \mathbf{B}|$ is equal to twice the area of the triangle whose coterminous sides are by the vectors **A** and **B**. (See Figure 19.) Equally saying, the magnitude $|\mathbf{A} \times \mathbf{B}|$ is equal to the area of the parallelogram spanned by **A** and **B**.

[9] Note that this property provides us with another verification for the fact, mentioned just above, that if two vectors are parallel, their vector product is the null vector.

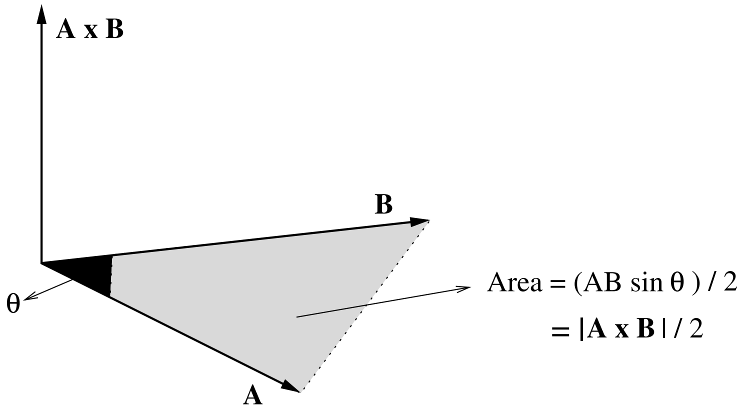

Figure 19: Area of the triangle = $\frac{1}{2}|\mathbf{A} \times \mathbf{B}|$.

- It immediately follows from condition (1) that the vector product of any two parallel vectors is zero. This leads to the conclusion that the vector product of a vector with itself is the *null vector*:

$$\boxed{\mathbf{A} \times \mathbf{A} = \mathbf{0}}$$

- We can write this conclusion for the standard basis vectors as

$$\boxed{\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}}$$

- Another set of identities for the standard basis vectors follows from condition (3):

$$\boxed{\begin{array}{ll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{i} = -\mathbf{k} \\ \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} \\ \mathbf{k} \times \mathbf{i} = \mathbf{j} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array}}$$

- Notice that the two preceding equation sets automatically satisfy condition (2). In order to remember easily these two sets, there is a good mnemonic in terms of the cyclic arrangement

$$\ldots\ \mathbf{i}\ \mathbf{j}\ \mathbf{k}\ \mathbf{i}\ \mathbf{j}\ \mathbf{k}\ \ldots$$

  The vector product of a vector (say, **i**) with its *right neighbor* (**j**) is the next following vector (**k**) when reading to the right; the vector product of a vector (**i**) with its *left neighbor* (**k**) is the next following vector (−**j**) when reading to the left.

- The components of the vector product $\mathbf{A} \times \mathbf{B}$ are deduced by using the two equation sets above as

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\,\mathbf{i} + (A_z B_x - A_x B_z)\,\mathbf{j} + (A_x B_y - A_y B_x)\,\mathbf{k}$$

This seemingly awkward equation is asymmetric and this makes it hard to remember. If we have a closer look at this equation and recall the evaluation of a 3×3 square matrix, we realize that the vector product $\mathbf{A} \times \mathbf{B}$ with its components can be easily deduced from the following determinant:

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \mathbf{j} \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \mathbf{k} \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \end{aligned}$$

A mnemonic to evaluate a 3×3 determinant in an easier manner is presented in Figure 20. First extend the matrix by repeating the first two columns to the right of the third column. The value of the determinant is the sum of the products corresponding to the three complete *right* diagonals minus the sum of the products corresponding to the three complete *left* diagonals:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + dhc + gbf - gec - ahf - dbi$$

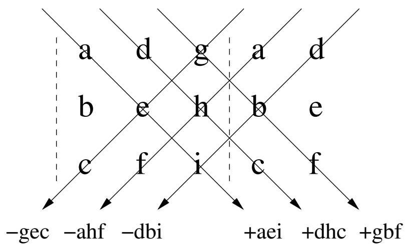

Figure 20: A mnemonic for the evaluation of a 3×3 square matrix.

- With the help of this mnemonic, it is no longer hard to deal with equations involving vector products. For instance, the following properties of the vector product, which we stated before, can now be easily verified:

$$\begin{aligned}
\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B}) &= \mathbf{B}\cdot(\mathbf{A}\times\mathbf{B}) = 0 \\
\mathbf{A}\times\mathbf{B} &= -\mathbf{B}\times\mathbf{A} \\
c\,(\mathbf{A}\times\mathbf{B}) &= (c\,\mathbf{A})\times\mathbf{B} = \mathbf{A}\times(c\,\mathbf{B}) \\
\mathbf{A}\times(\mathbf{B}+\mathbf{C}) &= \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C} \\
(\mathbf{A}+\mathbf{B})\times\mathbf{C} &= \mathbf{A}\times\mathbf{C}+\mathbf{B}\times\mathbf{C} \\
\mathbf{A}\times\mathbf{A} &= \mathbf{0}
\end{aligned}$$

Key Words, Phrases, and Equations

- Scalar quantity, vector quantity
- The null (zero) vector
- Equal vectors: $\mathbf{A} = \mathbf{B}$
- Resultant vector
- Commutativity, anticommutativity, associativity, distributivity
- (Vector) summation and subtraction:

$$\mathbf{A}\pm\mathbf{B} = (A_x \pm B_x)\,\mathbf{i} + (A_y \pm B_y)\,\mathbf{j} + (A_z \pm B_z)\,\mathbf{k}$$

- Scalar multiplication: $c\mathbf{A}$
- (Additive) inverse of $\mathbf{A}$: $-\mathbf{A}$
- Resolving (or decomposing) a vector into its components in an xy-plane:

$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y$$

$$A_x = A\cos\theta, \quad A_y = A\sin\theta$$

$$A = \sqrt{A_x^2 + A_y^2}$$

$$\tan\theta = \frac{A_y}{A_x} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$

- The component notation: (A_x, A_y)

- The magnitude-angle notation: (A, θ)
- The unit-vector notation: $\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$
- Unit vector: $\mathbf{A} = A\hat{\mathbf{A}}$ where $\hat{\mathbf{A}} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$
- The right-handed rectangular xyz-coordinate system
- The cartesian unit (or base) vectors: $(\mathbf{i}, \mathbf{j}, \mathbf{k})$
- The magnitude (or length) of a vector: $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$
- Position vector: $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
- Displacement vector: $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$
- Scalar (or dot or inner) product:

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$$

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$$

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = \mathbf{i} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{j} = 0$$

$$A = \sqrt{\mathbf{A} \cdot \mathbf{A}} \qquad \text{or} \qquad A^2 = \mathbf{A} \cdot \mathbf{A}$$

- Vector (or cross or outer) product:

$$|\mathbf{A} \times \mathbf{B}| = AB\sin\theta$$

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

$$\mathbf{A} \times \mathbf{A} = \mathbf{0}$$

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

$$\begin{array}{ll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \qquad \mathbf{j} \times \mathbf{i} = -\mathbf{k} \\ \mathbf{j} \times \mathbf{k} = \mathbf{i} & \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \\ \mathbf{k} \times \mathbf{i} = \mathbf{j} & \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array}$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

- The right-hand rule