# CHAPTER 0 | MATHEMATICS REVIEW

As I said earlier, the language of physics is calculus. This fact first requires that a serious engineering student be well acquainted as soon as possible with the rudiments of algebra, trigonometry, and geometry. To this end, here is a brief review of operations and methods.



## Mathematical Notation

- Here are the symbols that you will frequently come across throughout our course:

| Notation | Meaning |
|---|---|
| $y \equiv x$ | $y$ is **defined as**, or **identical to** $x$. |
| $y = x$ | $y$ is **equal to** $x$. |
| $y \neq x$ | $y$ is **not equal to** $x$. |
| $y \approx x$ | $y$ is **approximately equal to** $x$. |
| $y \sim x$ | $y$ is **in the order of magnitude of** $x$. |
| $y > x$ | $y$ is **greater than** $x$. |
| $y \gg x$ | $y$ is **much greater than** $x$. |
| $y < x$ | $y$ is **less than** $x$. |
| $y \ll x$ | $y$ is **much less than** $x$. |
| $y \geq x$ | $y$ is **greater than or equal to** $x$. |
| $y \leq x$ | $y$ is **less than or equal to** $x$. |
| $y \propto x$ | $y$ is **proportional to** $x$. |

- One of the very first step towards physics is to grasp firmly the meaning of $\Delta f$:

  > $\Delta f$ means the **change in the quantity** $f$, and defined by
  >
  > $$\Delta f = f_{\text{final}} - f_{\text{initial}}$$

  Here $f$ can be *any* physical quantity.

  For example, if $x_1$ is the initial position of a particle and $x_2$ is its final position, then the change in its position is written as

  $$\Delta x = x_2 - x_1.$$

- We shall say more about $\Delta f$ and its differential counterpart *df* in the class. Suffice it to say for the time being that $\Delta f$ implies a *measurable*, or *observable*, change in $f$; in case of *df*, on the other hand, the change in $f$ is *immeasurably small*, or *infinitesimally small*, but not zero.

- It is important to recall $|x|$, although it is already well-known by every one of you:

  > $|x|$ means the **absolute value of** $x$, which is *always positive*, regardless of the sign of $x$.

  For example, $|8| = 8$ and $|-3| = 3$.

- We will often use the Greek letter $\Sigma$ for the summation of several quantities. An example is

  $$m_1 + m_2 + m_3 + m_4 + m_5 \equiv \sum_{i=1}^{5} m_i.$$

# Basic Algebra

### Solving Mathematical Equations

- Throughout these freshman physics courses, you shall always solve a given or resulting mathematical equation for an **unknown**, which will be usually $x$, $y$, or $z$. First we state the golden rule:

  | Whatever you do on the left side of an equation, you must do the same on its right side. |
  |---|

  That is, you must always multiply or divide the both sides of the equation by the same number; similarly, the same quantity must be added to or subtracted from the both sides of the equation at the same time. The aim is to leave the unknown variable alone on one side.

### Examples

In the followings, we shall solve for the unknown $x$.

1. The equation to be solved:

$$2x + 7 = -6$$

   Add $-7$ to both sides:

$$2x + 7 - 7 = -6 - 7 \quad \Rightarrow \quad 2x = -13$$

   Divide both sides by 2:

$$\frac{2x}{2} = \frac{-13}{2} \quad \Rightarrow \quad x = \frac{-13}{2} \qquad \square$$

2. The equation to be solved:

$$5 = \frac{4-x}{x-2}$$

Multiply both sides by $(x-2)$:

$$5(x-2) = \frac{4-x}{x-2}\,(x-2) \quad \Rightarrow \quad 5x - 10 = 4 - x$$

Add 10 to both sides:

$$5x - 10 + 10 = 4 - x + 10 \quad \Rightarrow \quad 5x = 14 - x$$

Add $x$ to both sides:

$$5x + x = 14 - x + x \quad \Rightarrow \quad 6x = 14$$

Divide both sides by 6:

$$\frac{6x}{6} = \frac{14}{6} \quad \Rightarrow \quad x = \frac{14}{6} = \frac{7}{3} \qquad \square$$

3. The equation to be solved:

$$\frac{3a-5}{2x+3} = \frac{12}{7}$$

Multiply both sides by $(2x+3)$:

$$\frac{3a-5}{2x+3}\,(2x+3) = \frac{12}{7}\,(2x+3) \quad \Rightarrow \quad 3a - 5 = \frac{12}{7}\,(2x+3)$$

Multiply both sides by 7:

$$7\,(3a-5) = 12\,(2x+3) \quad \Rightarrow \quad 21a - 35 = 24x + 36$$

Add $-36$ to both sides:

$$21a - 35 - 36 = 24x + 36 - 36 \quad \Rightarrow \quad 21a - 71 = 24x$$

Divide both sides by 24:

$$\frac{21a-71}{24} = \frac{24x}{24} \quad \Rightarrow \quad x = \frac{21a-71}{24} \qquad \square$$

**Basic Rules**

- There should be no flaw at all in the following operations:

$$
\begin{aligned}
\text{Addition:} \quad & \frac{a}{b} \pm \frac{c}{d} &=& \frac{a \cdot d \pm b \cdot c}{b \cdot d} \\
\text{Multiplication:} \quad & \frac{a}{b} \cdot \frac{c}{d} &=& \frac{a \cdot c}{b \cdot d} \\
\text{Division:} \quad & \frac{a/b}{c/d} &=& \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}
\end{aligned}
$$

**Examples**

$$
\begin{aligned}
\frac{7}{8} + \frac{5}{6} &= \frac{(7)(6) + (5)(8)}{(8)(6)} = \frac{82}{48} = \frac{41}{24} \\
\frac{1}{5} - \frac{2}{3} &= \frac{(1)(3) - (2)(5)}{(3)(5)} = -\frac{7}{15} \\
\frac{1}{2} \cdot \frac{3}{4} &= \frac{(1)(3)}{(2)(4)} = \frac{3}{8} \\
\frac{3/7}{2/5} &= \frac{3}{7} \cdot \frac{5}{2} = \frac{(3)(5)}{(7)(2)} = \frac{15}{14}
\end{aligned}
$$

### Powers

- The followings are the rules of exponents:

$$
\begin{aligned}
x^0 &= 1\\
x^1 &= x\\
x^m x^n &= x^{m+n}\\
\frac{x^m}{x^n} &= x^{m-n}\\
x^{1/n} &= \sqrt[n]{x}\\
x^{m/n} &= \sqrt[n]{x^m}\\
(x^m)^n &= x^{m\cdot n}
\end{aligned}
$$

### Examples

$$
\begin{array}{ccc}
4^0 = 1, & (-2)^0 = 1, & -2^0 = -1\\
x^0 = 1, & 3.2^1 = 3.2, & x^1 = x\\
5^2 \cdot 5^6 = 5^8, & y^3 \cdot y^{-5} = y^{-2}, & 2^6/2^2 = 2^4\\
z^{-4}/z^3 = z^{-7}, & \left(6^3\right)^4 = 6^{12}, & \left(x^{-7}\right)^2 = x^{-14}\\
12^{1/4} = 1.8612, & 3^{1/5} = 1.2457, & 18.2^{-2/3} = 0.14453
\end{array}
$$

For the last three examples, just use a calculator.

**Factoring**

- The followings are frequently used factoring formulas:

$$\begin{aligned} \text{factoring by grouping:} \quad ax + ay + az &= a(x + y + z) \\ \text{perfect square trinomials:} \quad x^2 \pm 2xy + y^2 &= (x \pm y)^2 \\ \text{difference of two squares:} \quad x^2 - y^2 &= (x + y)(x - y) \\ \text{difference of two cubes:} \quad x^3 - y^3 &= (x - y)\left(x^2 + xy + y^2\right) \\ \text{sum of two cubes:} \quad x^3 + y^3 &= (x + y)\left(x^2 - xy + y^2\right) \end{aligned}$$

**Quadratic Equations**

- They are polynomial equations of the second degree, with the general form

  $$ax^2 + bx + c = 0$$

  where $x$ is unknown variable and $a$, $b$, and $c$ are the **coefficients** of the equation, with $a \neq 0$. In this course, and in others also, you will frequently come across quadratic equations; so it's imperative that a diligent engineering student be able to find easily the **roots** of any given quadratic equation by using

  $$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

  Here if $b^2 - 4ac \geq 0$, the roots are *real*, and if $b^2 - 4ac < 0$, they are **complex**. You will not encounter the latter in our course.

- Note that a quadratic equation with its roots $x_1$ and $x_2$ can be factored out as

  $$ax^2 + bx + c = 0 \quad \Rightarrow \quad \text{const.}\,(x - x_1)\,(x - x_2) = 0$$

  where "const." is *any* constant.

**Examples**

1. For the roots of the quadratic equation

$$2x^2 + x - 21 = 0$$

we first note that $a = 2$, $b = 1$, and $c = -21$, so that

$$x_{1,2} = \frac{-1 \pm \sqrt{1^2 - (4)(2)(-21)}}{(2)(2)} = \frac{-1 \pm \sqrt{169}}{4} = \frac{-1 \pm 13}{4}$$

whence

$$x_1 = 3 \quad \text{and} \quad x_2 = -\tfrac{7}{2} \qquad \square$$

2. For the equation

$$x^2 - 3x - 5 = 0$$

the roots are found as

$$x_{1,2} = \frac{-(-3) \pm \sqrt{(-3)^2 - (4)(1)(-5)}}{(2)(1)} = \frac{3 \pm \sqrt{29}}{2}$$

and

$$x_1 = \frac{3 + \sqrt{29}}{2} = 4.1926 \quad \text{and} \quad x_2 = \frac{3 - \sqrt{29}}{2} = -1.1926 \qquad \square$$

**Linear Equations**

- The general form of a **linear equation** is

$$\boxed{y = mx + b}$$

where $m$ and $b$ are constants.

- We refer to this equation as being *linear* since the graph of $y$ versus $x$ is a straight line.

- The constant $b$, called the **intercept**, represents the value of $y$ at which the straight line intersects the $y$-axis.

- The constant $m$ is equal to the **slope** of the straight line and is also equal to the tangent of the angle that the line makes with the $x$-axis. If any two points on the straight line are specified by the coordinates $(x_1, y_1)$ and $(x_2, y_2)$, then the slope of the straight line is

$$\text{slope} \equiv \tan\theta = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where if $a > 0$ ($a < 0$), the straight line has a *positive* (*negative*) slope.

### Solving Simultaneous Linear Equations

- Keep always in your mind the following fact:

> In general, if a problem has $n$ unknowns, its solution requires (at least) $n$ equations.

- In order to solve two simultaneous equations involving two unknowns, $x$ and $y$, we solve one of the equations for $x$ in terms of $y$ and substitute this expression into other equation.

- Two linear equations with two unknowns can also be solved by a graphical method. If the straight lines corresponding to the two equations are plotted in a conventional coordinate system, the intersection of the two lines represents the solution.

### Logarithms

- Suppose that a quantity $x$ is expressed as a power of some quantity $a$: $x = a^y$. The number $a$ is called the **base** number. The **logarithm** of $x$ with respect to the base $a$ is equal to the exponent to which the base must be raised in order to satisfy the expression $x = a^y$. The rule is written as

$$x = a^y \quad \Leftrightarrow \quad y = \log_a x$$

- In practice, the two bases most often used are the base 10, called the **common logarithm base**, and base $e$, called the **natural logarithm base**:

$$x = 10^y \quad \Leftrightarrow \quad y = \log_{10} x$$

$$x = e^y \quad \Leftrightarrow \quad y = \ln x$$

$$e = 2.7182818284...$$

- Some useful properties of logarithms are:

$$\begin{aligned} \log(ab) &= \log a + \log b \\ \log(a/b) &= \log a - \log b \\ \log(a^n) &= n \log a \\ \ln e &= 1 \\ \ln e^a &= a \\ \ln(1/a) &= -\ln a \end{aligned}$$

## Geometry

- The **distance** $d$ between two points whose coordinates are $(x_1, y_1)$ and $(x_2, y_2)$ is calculated via

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- The **radian measure**: the arc length $s$ of a circular arc is proportional to the radius $r$ for a fixed value of $\theta$ in radians (Fig 1).

$$\boxed{s = r\theta}$$

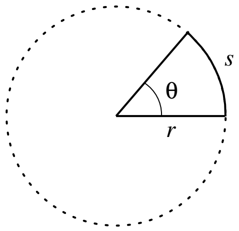


Figure 1: Arc length $s$.

- For a **rectangle** ($\ell$: length, $w$: width)

$$\boxed{\text{Area} = \ell w}$$

- For a **triangle** ($b$: base, $h$: height)

$$\boxed{\text{Area} = \frac{1}{2}\, bh}$$

- For a **rectangular box** ($\ell$: length, $w$: width, $h$: height)

$$\begin{aligned} \text{Area} &= 2(\ell h + \ell w + hw) \\ \text{Volume} &= \ell wh \end{aligned}$$

- For a **circle** ($r$: radius)

$$\begin{aligned} \text{Circumference} &= 2\pi r \\ \text{Area} &= \pi r^2 \end{aligned}$$

- For a **sphere** ($r$: radius)

$$\begin{aligned} \text{Surface area} &= 4\pi r^2 \\ \text{Volume} &= \tfrac{4}{3}\pi r^3 \end{aligned}$$

- For a **cylinder** ($r$: radius, $\ell$: length)

$$\begin{aligned} \text{Surface area} &= 2\pi r^2 + 2\pi r\ell \\ \text{Volume} &= \pi r^2 \ell \end{aligned}$$

- The equation of a **straight line**, as in Fig. 2, is given by

$$y = mx + b$$

where $b$ is the $y$-intercept and $m$ is the slope of the line.

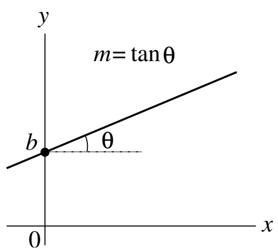


Figure 2: A straight line.

- The equation of a **circle** of radius $r$ centered at the origin is

$$x^2 + y^2 = r^2$$

- The equation of an **ellipse**, as in Fig. 3, with the origin at its center is

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

where $a$ ($b$) is the length of the semi-major (semi-minor) axis.

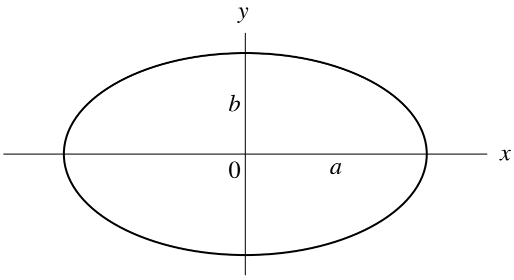


Figure 3: An ellipse.

- The equation of a **parabola**, as in Fig. 4, whose vertex is at $y = b$ is given by

$$\boxed{y = ax^2 + b}$$

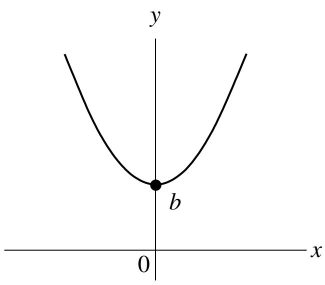


Figure 4: A parabola.

- The equation of a **rectangular hyperbola**, as in Fig. 5, is

$$xy = \text{constant}$$

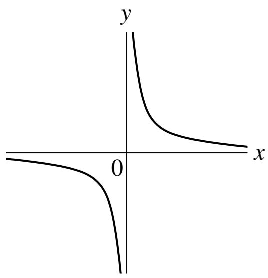


Figure 5: A hyperbola.

## Trigonometry

- That portion of mathematics based on the special properties of the right angle is called **trigonometry**.
- By definition, a **right triangle**, as in Fig. 6, is one containing a 90°-angle.

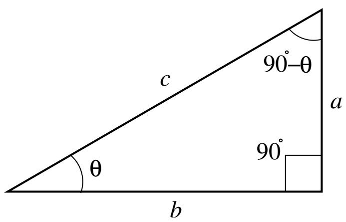


Figure 6: A right triangle.

- The **Pythagorean theorem** provides the following relationship between the sides of the right triangle shown in Fig. 6:

$$c^2 = a^2 + b^2$$

- The most important trigonometric expression follows from the Pythagorean theorem as

$$\sin^2\theta + \cos^2\theta = 1$$

- The three basic trigonometric functions defined via the right triangle shown in Fig. 6 are the **sine** (sin), **cosine** (cos), and the **tangent** (tan) functions:

$$\begin{aligned}
\sin\theta &\equiv \frac{\text{side opposite }\theta}{\text{hypotenuse}} = \frac{a}{c} \\
\cos\theta &\equiv \frac{\text{side adjacent }\theta}{\text{hypotenuse}} = \frac{b}{c} \\
\tan\theta &\equiv \frac{\text{side opposite }\theta}{\text{side adjacent }\theta} = \frac{a}{b}
\end{aligned}$$

- The **cosecant** (csc), **secant** (sec), **tangent** (tan), and **cotangent** (cot) functions are defined as

$$\begin{aligned}
\csc\theta &\equiv \frac{1}{\sin\theta} \\
\sec\theta &\equiv \frac{1}{\cos\theta} \\
\tan\theta &\equiv \frac{\sin\theta}{\cos\theta} \\
\cot\theta &\equiv \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}
\end{aligned}$$

- The relations below follow directly from any right triangle, like the one in Fig. 6:

$$\begin{aligned}
\sin\theta &= \cos(90^\circ - \theta) \\
\cos\theta &= \sin(90^\circ - \theta) \\
\cot\theta &= \tan(90^\circ - \theta)
\end{aligned}$$

- Some properties of trigonometric functions are as follows:

$$
\begin{aligned}
\sin(-\theta) &= -\sin\theta \\
\cos(-\theta) &= \cos\theta \\
\tan(-\theta) &= -\tan\theta
\end{aligned}
$$

- Another three important trigonometric expressions are as follows:

$$
\begin{aligned}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\end{aligned}
$$

- For *any* triangle (Fig. 7), we have

$$
\begin{aligned}
\alpha + \beta + \gamma &= 180^\circ \\
\theta &= \beta + \gamma
\end{aligned}
$$

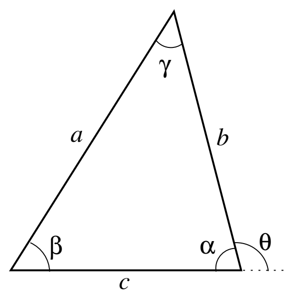


Figure 7: An ordinary triangle.