CHAPTER 4 | MOTION IN TWO AND THREE DIMENSIONS

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OpenCourseWare Project
PHYS 101

April 2013

Outline

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Position

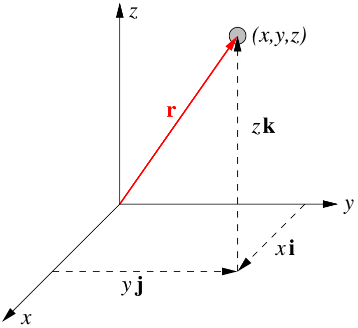

Figure: A position vector locating an object at a point (x, y, z).

Position vector:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \qquad \text{(m)}$$

$$r \equiv |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} \qquad \text{(m)}$$

Example

Question: Find the distance between the object and the origin.

Solution:

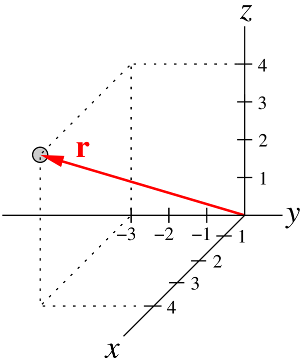

Figure: The position vector $\mathbf{r} = 4\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ for an object at point $(4, -3, 4)$.

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Displacement

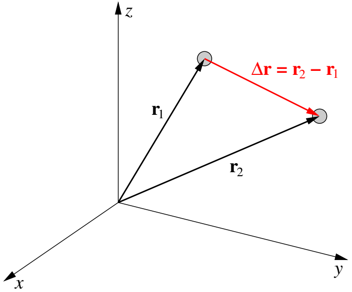

Figure: The displacement vector $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$.

Displacement vector:

$$\begin{aligned}\Delta\mathbf{r} &= \mathbf{r}_2 - \mathbf{r}_1 \\ &= (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k} \qquad \text{(m)} \\ &= \Delta x\,\mathbf{i} + \Delta y\,\mathbf{j} + \Delta z\,\mathbf{k}\end{aligned}$$

Example

Question: Let the object in the preceding example starts to move from the initial position $\mathbf{r}_1 = (4\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$ m, follows *some* path, and arrives at the final position $\mathbf{r}_2 = (3\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$ m at a later time. Find $\Delta\mathbf{r}$ and Δr.

Solution:

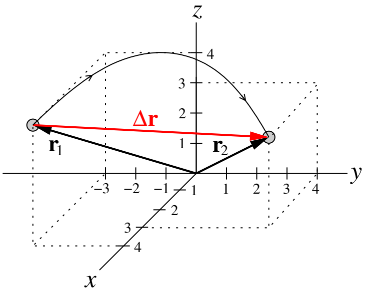

Figure: The displacement vector $\Delta\mathbf{r}$ for a moving object.

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Average Velocity

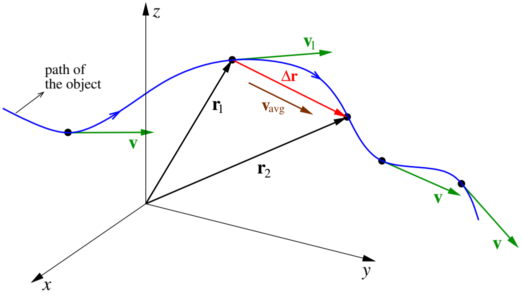

Figure: The average and instantaneous velocity vectors for a moving object.

Average velocity:

$$\mathbf{v}_{avg} = \frac{\Delta \mathbf{r}}{\Delta t} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{t_2 - t_1} \qquad \text{(m/s)}$$

Example

Question: An object changes its position according to the relation

$$\mathbf{r}(t) = (t + 1)\,\mathbf{i} + \left(t^2 - 4\right)\mathbf{j} - t\,\mathbf{k}$$

where r is in meters and t is in seconds. Find its average velocity between $t = 1$ s and $t = 3$ s.

Solution:

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(Instantaneous) Velocity

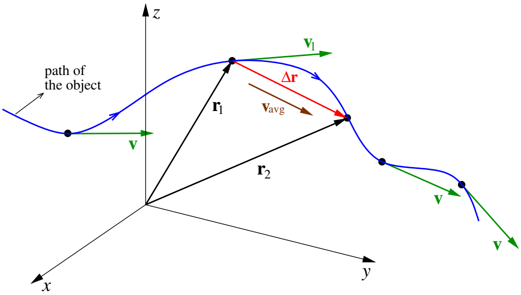

Figure: The average and instantaneous velocity vectors for a moving object.

$$\begin{aligned}\mathbf{v} &= \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt} \\ &= \mathbf{v}_x + \mathbf{v}_y + \mathbf{v}_z \\ &= v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k} \quad \text{(m/s)} \\ &= \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}\end{aligned}$$

speed: $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$

Example

Question: Let the position of an object be $\mathbf{r}(t) = (t+1)\,\mathbf{i} + \left(t^2 - 4\right)\mathbf{j} - t\,\mathbf{k}$ with r being in meters and t in seconds. Determine the general relation for the velocity of this object. Using this, find the object's velocities and speeds at $t = 1$ s and $t = 3$ s.

Solution:

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Average Acceleration

Average Acceleration

$$\mathbf{a}_{avg} = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{t_2 - t_1} \qquad \left(\mathrm{m/s^2}\right)$$

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(Instantaneous) Acceleration

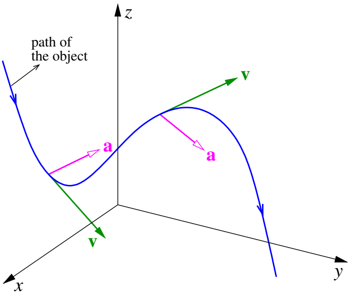

Figure: The instantaneous acceleration vector for a moving object.

$$
\begin{aligned}
\mathbf{a} &= \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \\
&= \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z \\
&= a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k} \quad \left(\mathrm{m/s^2}\right) \\
&= \frac{dv_x}{dt}\,\mathbf{i} + \frac{dv_y}{dt}\,\mathbf{j} + \frac{dv_z}{dt}\,\mathbf{k} \\
&= \frac{dx^2}{dt^2}\,\mathbf{i} + \frac{dy^2}{dt^2}\,\mathbf{j} + \frac{dz^2}{dt^2}\,\mathbf{k}
\end{aligned}
$$

Magnitude:

$$
a = \sqrt{a_x^2 + a_y^2 + a_z^2} \quad \left(\mathrm{m/s^2}\right)
$$

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Tangential and Radial Accelerations

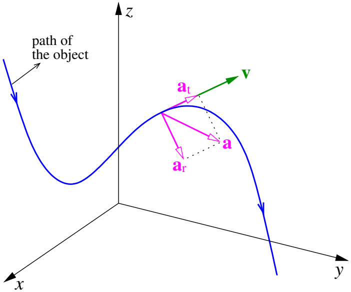

Figure: An acceleration vector **a** decomposed its tangential, $\mathbf{a}_t$, and radial, $\mathbf{a}_r$, components.

Tangential Acceleration

$$a_t = \frac{dv}{dt}$$

Radial Acceleration

$$a_r = \frac{v^2}{r}$$

Example

Question: The position of the bus is

$$\mathbf{r}(t) = \left(-\frac{1}{10}\,t^3 + 2t^2 - 10t\right)\mathbf{i} + \left(-t^2 + 2t + 12\right)\mathbf{j}$$

where r is in kilometers and t is in hours. Find the kinematic quantities of the bus at $t = 2$ h.

Solution:

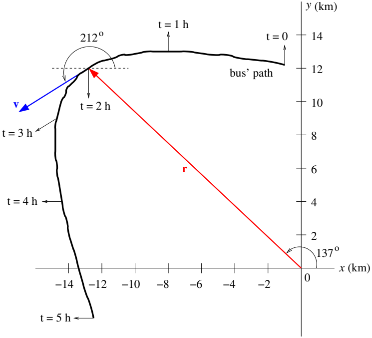

Figure: The position and velocity vectors of the bus at $t = 2$ h.

Example (cont.'d)

Solution (cont.'d):

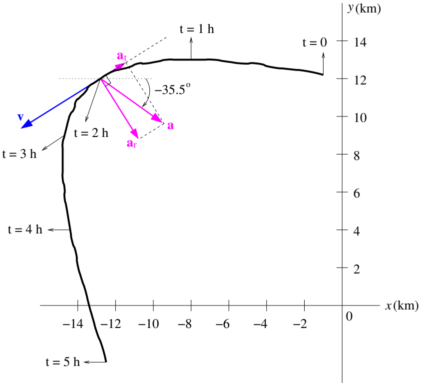

Figure: The velocity and acceleration vectors of the bus at $t = 2$ h.

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Motion with Constant Acceleration: 3D Consideration

Acceleration, Velocity, and Position Vectors

$$\begin{aligned} \mathbf{a} &= \text{const.} \\ \mathbf{v} &= \mathbf{v}_0 + \mathbf{a}t \\ \mathbf{r} &= \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2 \end{aligned}$$

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Motion with Constant Acceleration in a Plane: Projectile Motion

Components of Acceleration, Velocity, and Position Vectors

$$a_x = 0 \qquad a_y = -g$$

$$v_x = v_{x0} \qquad v_y = v_{y0} - gt$$

$$x = x_0 + v_{x0} t \qquad y = y_0 + v_{y0} t - \frac{1}{2} g t^2$$

$$v_y^2 = v_{y0}^2 - 2g\left(y - y_0\right)$$

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Example: Symmetric Projectile Motion

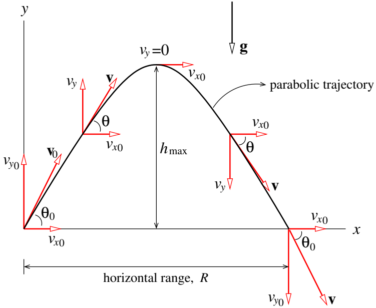


Figure: A symmetric projectile motion.

Parabolic Trajectory

$$y = (\tan\theta_0)\, x - \left(\frac{g}{2v_0^2\cos^2\theta_0}\right) x^2$$

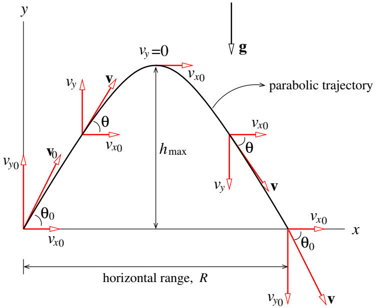
Maximum Altitude

$$h_{max} = \frac{v_0^2 \sin^2\theta_0}{2g}$$

Horizontal Range

$$R = \frac{v_0^2}{g}\, \sin 2\theta_0$$

Derivations of y, h_{max}, and R



Example: Symmetric Projectile Motion(cont.'d)

Complementary Angles (Example)

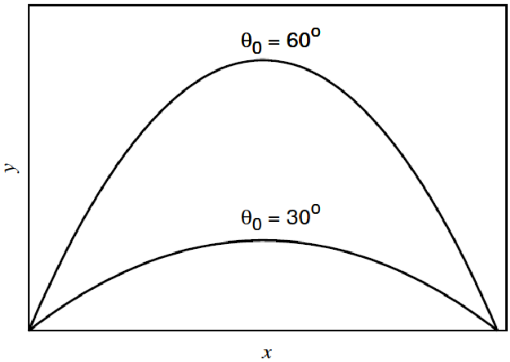


Figure: Two complementary initial angles lead to the same horizontal range *R* for a symmetric projectile motion.

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Uniform Circular Motion

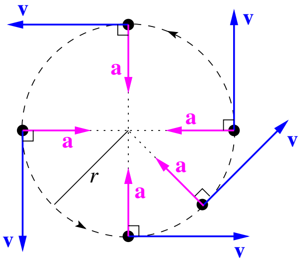

Figure: An object undergoing a uniform circular motion.

Radial (centripetal) acceleration

$$a = a_r = \frac{v^2}{r}$$

Period of a uniform circular motion

$$T = \frac{2\pi r}{v} \qquad \text{(s)}$$

Uniform Circular Motion (cont.'d)

Derivation of $a = a_\mathrm{r} = v^2/r$:

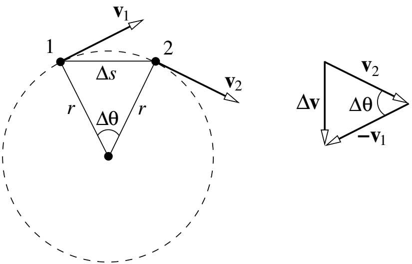

Figure: Figures for the derivation of $a = a_\mathrm{r} = v^2/r$.