CHAPTER 4 | MOTION IN TWO AND THREE DIMENSIONS

Introduction

- In Chapter 2 we had considered 1D motion along a straight line, and in the next chapter we had diverted our attention to vectors. Armed with the power of vectors, we resume in the present chapter analyzing motion, but this time we will focus on motion in two and three dimensions, i.e., 2D or 3D motion.
- As you shall see shortly, 2D and 3D motions are essentially the straightforward extension of 1D motion; that is, if you have grasped firmly the notions of 1D motion, you will have no difficulty at all in understanding 2D or 3D motion.
- Although the title of this chapter reads "motion in two and three dimensions," we shall limit our discussion mainly to the 2D case: motion in a *plane*.

Position

- We will show the position of a moving object at any time t by the so-called **position vector r**, whose tail is at the origin and whose head is at a point (x, y, z) (See Figure 1).

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad \text{(m)}$$

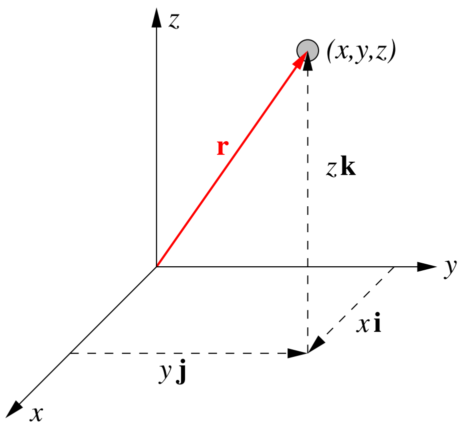

Figure 1: A position vector locating an object at a point (x, y, z).

where $x(t)$, $y(t)$, and $z(t)$ are the respective x-, y-, and x-coordinates of the object at time t.

- Out of visual clarity concern, I will omit time t from now on and simply write, for example, x instead of writing $x(t)$. With this in mind, I will rewrite the above equation for the position vector as

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \qquad \text{(m)}$$

The thumbrule you should observe is that $\mathbf{r}_0$ means, for example, the position vector of an object at time $t = 0$. Although we can use any of the notations $\mathbf{r}_0$, $\mathbf{r}(0)$, $\mathbf{r}(t = 0)$, $\mathbf{r}(t)|_{t=0}$ for this, I shall always use the first one, $\mathbf{r}_0$, and also recommend it to you. Now with this convention at hand, $\mathbf{r}$ means the position of the object *at any time* t, including $t = 0$. Similarly, y_0 will be the y-coordinate of the object at time $t = 0$, and y will mean its y-coordinate at any time t, including $t = 0$.

- It is now obvious that the magnitude of a position vector, which is the distance between the object and the origin, is found using

$$r \equiv |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} \qquad \text{(m)}$$

Example

In Figure 2 is the position vector $\mathbf{r} = 4\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ for an object located at point $(4, -3, 4)$. The length of this vector, i.e., the distance between the object and the origin, is

$$r = \sqrt{4^2 + (-3)^2 + 4^2} \text{ units} = \sqrt{41} \text{ units} \approx 6.40 \text{ units.}$$

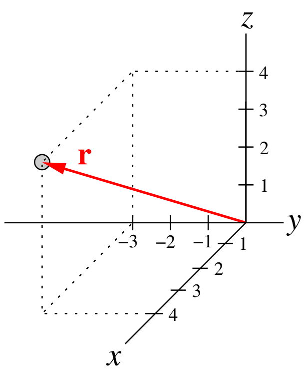

Figure 2: The position vector $\mathbf{r} = 4\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ for an object at point $(4, -3, 4)$.

Displacement

- Since a position vector $\mathbf{r}$ locates an object, when the particle changes its location, so does $\mathbf{r}$. As in Figure 3, if the object moves from position $\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and arrives at position $\mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$ at a later time, its **displacement vector**, denoted by $\Delta\mathbf{r}$, gives us the *net change* (or *overall change*) in its position. It is obvious that $\Delta\mathbf{r}$ equals the vector $\mathbf{r}_2 - \mathbf{r}_1$:

$$\begin{aligned} \Delta\mathbf{r} &= \mathbf{r}_2 - \mathbf{r}_1 \\ &= (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k} \qquad \text{(m)} \\ &= \Delta x\,\mathbf{i} + \Delta y\,\mathbf{j} + \Delta z\,\mathbf{k} \end{aligned}$$

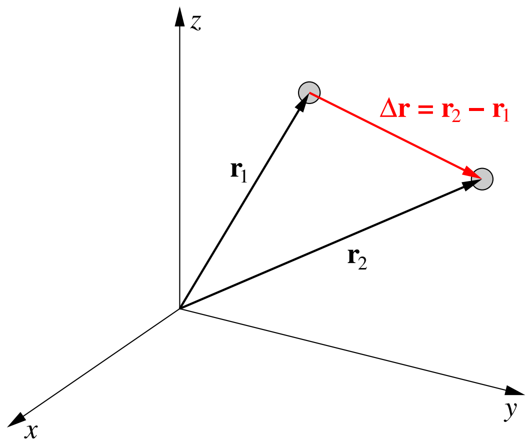

Figure 3: The displacement vector $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$.

- The following consideration is much more meaningful: if the object initially at $\mathbf{r}_1$ undergoes a displacement $\Delta\mathbf{r}$ and arrives at position $\mathbf{r}_2$, we must have

$$\boxed{\mathbf{r}_1 + \Delta\mathbf{r} = \mathbf{r}_2}$$

Example

Let the object in the preceding example starts to move from the initial position $\mathbf{r}_1 = (4\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$ m, follows *some* path, and arrives at the final position $\mathbf{r}_2 = (3\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$ m at a later time. Figure 4 illustrates the motion of the object.

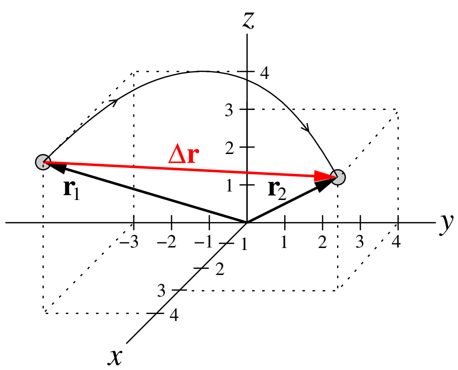

Figure 4: The displacement vector $\Delta\mathbf{r}$ for a moving object.

The displacement that the object experiences is

$$\begin{aligned}\Delta\mathbf{r} &= \mathbf{r}_2 - \mathbf{r}_1 \\ &= (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k} \\ &= (3-4)\,\mathbf{i} + (4-(-3))\,\mathbf{j} + (3-4)\,\mathbf{k} \\ &= (-\mathbf{i} + 7\mathbf{j} - \mathbf{k})\ \text{m}.\end{aligned}$$

The **distance** between the initial and final positions is the magnitude of the displacement vector we found above:

$$\Delta r = \sqrt{(-1)^2 + 7^2 + (-1)^2}\ \text{m} = \sqrt{51}\ \text{m} \approx 7.14\ \text{m}.$$

You should note that this distance is *not* equal to the length of the path that the object followed. The conclusion, therefore, is that the displacement vector of an object does *not* give any information about the actual path of the object; it does provide only the *net* change in the position of the object moving between two *definite* positions, $\mathbf{r}_1$ and $\mathbf{r}_2$.

Average Velocity

- Let a moving object be at position $\mathbf{r}_1$ at time t_1 and at position $\mathbf{r}_2$ at a later time t_2. The **average velocity** of the object, which is a vector quantity, is defined as

$$\boxed{\mathbf{v}_{\text{avg}} = \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{t_2 - t_1} \qquad (\text{m/s})}$$

- Average velocity of an object gives a rough answer to the question *how fast* the object moves between the two positions $\mathbf{r}_1$ and $\mathbf{r}_2$.

- Since $t_2 > t_1$, Δt is always a positive quantity. It thus follows from the above equation that the average velocity vector $\mathbf{v}_{\text{avg}}$ is in the direction of the displacement vector $\Delta\mathbf{r}$. Therefore, $\mathbf{v}_{\text{avg}}$ and $\Delta\mathbf{r}$ in Figure 5 below are drawn *in parallel*.

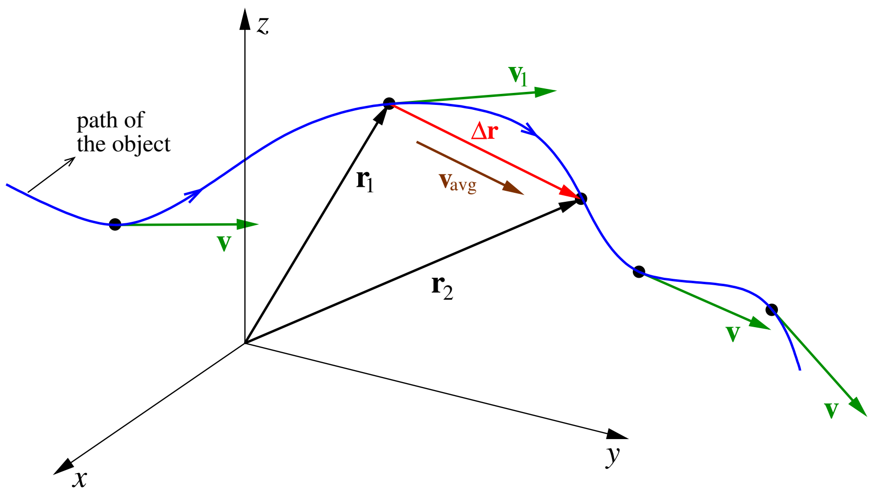

Figure 5: The average and instantaneous velocity vectors for a moving object.

Example

An object changes its position according to the relation

$$\mathbf{r}(t) = (t+1)\,\mathbf{i} + \left(t^2 - 4\right)\mathbf{j} - t\,\mathbf{k}$$

where r is in meters and t is in seconds. Let us find its average velocity between $t = 1$ s and $t = 3$ s. We need first the positions at these times:

$$\begin{aligned}
\mathbf{r}(1) &= (1+1)\,\mathbf{i} + \left(1^2 - 4\right)\mathbf{j} - 1\,\mathbf{k} = (2\,\mathbf{i} - 3\,\mathbf{j} - \mathbf{k})\ \text{m} \\
\mathbf{r}(3) &= (3+1)\,\mathbf{i} + \left(3^2 - 4\right)\mathbf{j} - 3\,\mathbf{k} = (4\,\mathbf{i} - 5\,\mathbf{j} - 3\,\mathbf{k})\ \text{m}
\end{aligned}$$

With these, the desired average velocity is found as

$$\begin{aligned}
\mathbf{v}_{\text{avg}} &= \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\mathbf{r}(3) - \mathbf{r}(1)}{3-1} = \frac{(4\,\mathbf{i} - 5\,\mathbf{j} - 3\,\mathbf{k}) - (2\,\mathbf{i} - 3\,\mathbf{j} - \mathbf{k})}{2} \\
&= (\mathbf{i} - \mathbf{j} - \mathbf{k})\ \text{m/s}
\end{aligned}$$

and its magnitude is

$$v_{\text{avg}} = \sqrt{1^2 + (-1)^2 + (-1)^2} = \sqrt{3}\ \text{m/s} \approx 1.73\ \text{m/s}.$$

(Instantaneous) Velocity

- Since $\mathbf{v}_{\text{avg}}$ is not sufficiently informative, most of the time we are interested in the **(instantaneous) velocity v** of an object at some instant. We define $\mathbf{v}$ as the limiting value $\mathbf{v}_{\text{avg}}$ as the time interval Δt approaches zero:

$$\boxed{\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt} \qquad \text{(m/s)}}$$

- Saying every time "instantaneous velocity" is somewhat cumbersome; henceforth we will say just **velocity v**, without the redundant adjective "instantenous."

- The above definition implies implicitly that *the velocity* $\mathbf{v}$ *of a moving object is everywhere tangent to the object's path*, as illustrated in Figure 5. This is an important piece of information; you'd better know it by heart.

- Using the explicit form of the position vector, $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, the above relation for the velocity $\mathbf{v}$ leads to

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}$$

Here it is natural to interpret dx/dt as the magnitude of the velocity vector in the x-direction; the similar interpretations follow also for dy/dt and dz/dt:

$$v_x = \frac{dx}{dt}, \qquad v_y = \frac{dy}{dt}, \qquad v_z = \frac{dz}{dt}$$

We can thus write the velocity vector in terms of rectangular coordinates in three different forms as

$$\boxed{\begin{aligned} \mathbf{v} &= \mathbf{v}_x + \mathbf{v}_y + \mathbf{v}_z \\ &= v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k} \\ &= \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k} \end{aligned}}$$

- The magnitude of the velocity vector of an object at a certain time is called its **speed** at that time:

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

Note that speed is always a positive quantity.

Example

Let's return to the previous example where the position of an object was given as

$$\mathbf{r}(t) = (t+1)\,\mathbf{i} + \left(t^2 - 4\right)\mathbf{j} - t\,\mathbf{k}$$

with r being in meters and t in seconds. The general relation for the velocity of this object is determined as

$$\begin{aligned}\mathbf{v}(t) &= \frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left[(t+1)\,\mathbf{i} + \left(t^2 - 4\right)\mathbf{j} - t\,\mathbf{k}\right] \\ &= (\mathbf{i} + 2t\,\mathbf{j} - \mathbf{k})\ \text{m/s}\end{aligned}$$

Using this, we can now find the object's velocity at $t = 1$ s and $t = 3$ s:

$$\mathbf{v}(1) = (\mathbf{i} + 2\mathbf{j} - \mathbf{k})\ \text{m/s} \quad \text{and} \quad \mathbf{v}(3) = (\mathbf{i} + 6\mathbf{j} - \mathbf{k})\ \text{m/s}$$

The corresponding speeds are calculated to be

$$v(1) = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}\ \text{m/s} \approx 2.45\ \text{m/s}$$

and

$$v(3) = \sqrt{1^2 + 6^2 + (-1)^2} = \sqrt{38}\ \text{m/s} \approx 6.16\ \text{m/s}$$

In the previous example, we had found the magnitude of the object's average velocity between $t = 1$ s and $t = 3$ s as $v_{\text{avg}} \approx 1.73$ m/s. This result is *not* equal at all to $\frac{1}{2}\left[v(1) + v(3)\right] \approx 4.31$ m/s, as might have been *wrongly* guessed.

Average Acceleration

- For an object whose velocity at time t_1 is $\mathbf{v}_1$ and at a later time t_2 is $\mathbf{v}_2$, we define its **average acceleration** $\mathbf{a}_{\text{avg}}$, which is a another vector quantity, as

$$\boxed{\mathbf{a}_{\text{avg}} = \frac{\Delta\mathbf{v}}{\Delta t} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{t_2 - t_1} \qquad \left(\text{m/s}^2\right)}$$

(Instantaneous) Acceleration

- The **(instantaneous) acceleration a** of an object at any time is defined as the limiting value $\mathbf{a}_{\text{avg}}$ as Δt goes zero:

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt} \qquad (\text{m/s}^2)$$

  That is, the acceleration vector **a** is the first time-derivative of the velocity vector **v**.

- Since the acceleration vector **a** is a measure of the change of the velocity vector **v** at some instant t, the direction of **a** in regard to **v** gives two important information at t: (1) in which way the path of the object in question curves, as it is illustrates in Figure 6, and (2) if the object's *speed* is increasing or decreasing. In other words, if an object has a non-zero acceleration vector, its velocity vector must change either its direction or its magnitude, or both.

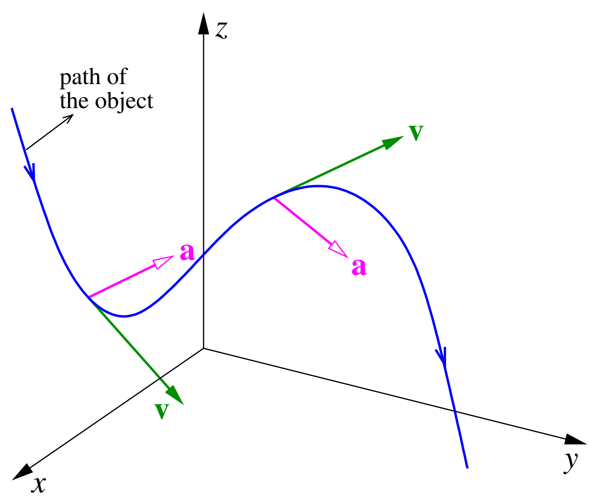

Figure 6: The average and instantaneous velocity vectors for a moving object.

- We can decompose the acceleration vector into two components, as in Figure 7; one of them is parallel (or antiparallel) to the velocity vector **v** (or tangent to the path of the object) and is called the **tangential acceleration** $\mathbf{a}_{\text{t}}$; the other one is perpendicular to **v** and is referred to as **radial** (or, **centripetal**) **acceleration** $\mathbf{a}_{\text{r}}$. Now the rule that you should know by heart is stated as:

The tangential acceleration $\mathbf{a}_\mathrm{t}$ is responsible *only* for the change in the magnitude of the object's velocity (i.e., for its *speed*); the direction of its velocity is governed only by the radial acceleration $\mathbf{a}_\mathrm{r}$.

Mathematically, these two statements amount respectively to

$$a_\mathrm{t} = \frac{dv}{dt} \qquad \text{and} \qquad a_\mathrm{r} = \frac{v^2}{r}$$

where r is the radius of the curvature of the path at the point where the radial acceleration is calculated. The first formula for a_t here should be obvious from the above discussion. The second one for a_r is subtle for the time being, but will be clear before the end of this chapter.

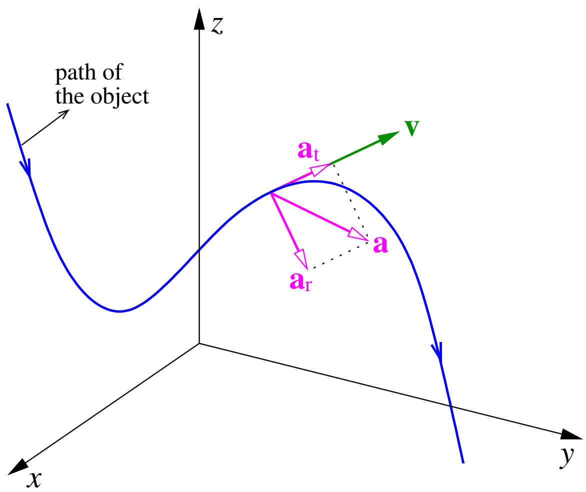

Figure 7: An acceleration vector $\mathbf{a}$ decomposed its tangential, $\mathbf{a}_\mathrm{t}$, and radial, $\mathbf{a}_\mathrm{r}$, components.

- Since $\mathbf{v} = d\mathbf{r}/dt$, $\mathbf{a}$ is seen to be equal also the second time-derivative of the position vector $\mathbf{r}$:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \qquad \left(\mathrm{m/s^2}\right)$$

- Since $\mathbf{v} = v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k}$, the acceleration can be written in the unit-vector notation as

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k}\right) = \frac{dv_x}{dt}\,\mathbf{i} + \frac{dv_y}{dt}\,\mathbf{j} + \frac{dv_z}{dt}\,\mathbf{k}$$

where dv_x/dt is the magnitude of the acceleration vector in the x-direction, and so on:

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}, \qquad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}, \qquad a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}$$

With these, the acceleration of an object in rectangular coordinates can be expressed in four different forms:

$$\begin{aligned}
\mathbf{a} &= \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z \\
&= a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k} \\
&= \frac{dv_x}{dt}\,\mathbf{i} + \frac{dv_y}{dt}\,\mathbf{j} + \frac{dv_z}{dt}\,\mathbf{k} \\
&= \frac{dx^2}{dt^2}\,\mathbf{i} + \frac{dy^2}{dt^2}\,\mathbf{j} + \frac{dz^2}{dt^2}\,\mathbf{k}
\end{aligned}$$

- The magnitude of the acceleration vector of an object at some instant is

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

which has *no* special name as that of the velocity vector, which was *speed*.

Example

The position of a bus, of old days, around a high mountain, which is shown in Figure 8, could be approximated by

$$\mathbf{r}(t) = \left(-\tfrac{1}{10}\,t^3 + 2t^2 - 10t\right)\mathbf{i} + \left(-t^2 + 2t + 12\right)\mathbf{j}$$

where r is in kilometers and t is in hours. Let us find the kinematic quantities of the bus at $t = 2$ h. To this end, we first determine the velocity and acceleration vectors:

$$\begin{aligned}
\mathbf{v}(t) &= \frac{d\mathbf{r}}{dt} = \left(-\tfrac{3}{10}\,t^2 + 4t - 10\right)\mathbf{i} + (-2t + 2)\,\mathbf{j} \\
\mathbf{a}(t) &= \frac{d\mathbf{v}}{dt} = \left(-\tfrac{3}{5}\,t + 4\right)\mathbf{i} - 2\,\mathbf{j}
\end{aligned}$$

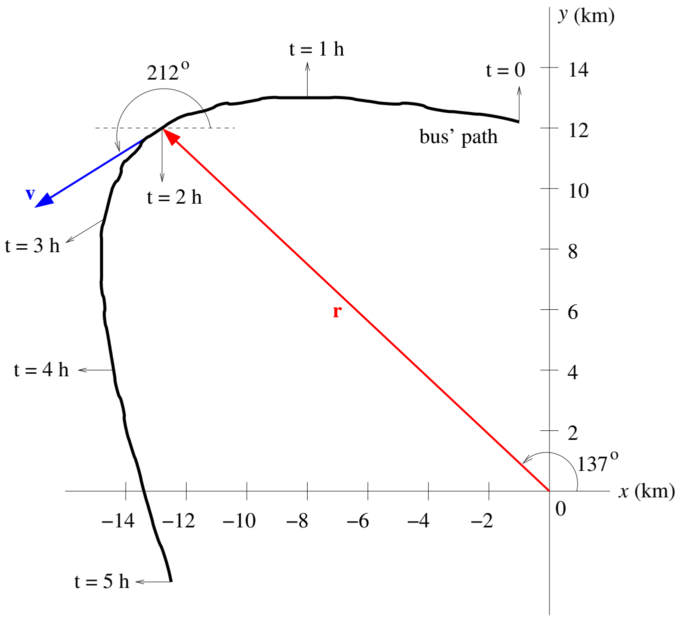

Figure 8: The position and velocity vectors of the bus at $t = 2$ h.

The position of the bus *in unit-vector notation* at $t = 2$ h is

$$\begin{aligned} \mathbf{r}(2) &= \left(-\tfrac{1}{10}\,2^3 + 2\cdot 2^2 - 10\cdot 2\right)\mathbf{i} + \left(-2^2 + 2\cdot 2 + 12\right)\mathbf{j} \\ &= (-12.8\,\mathbf{i} + 12.0\,\mathbf{j})\ \text{km}, \end{aligned}$$

or we have *in magnitude-angle notation* that

$$r(2) = \sqrt{(-12.8)^2 + 12.0^2} \approx 17.5\ \text{km},$$

and

$$\alpha = \arctan\left(\frac{12.0}{-12.8}\right) = -43.152^\circ + 180^\circ \approx 137^\circ$$

We thus say that the bus is at a distance of 17.5 km and makes an angle of 137° with respect to the positive x-axis. The position vector $\mathbf{r}(2)$ is shown in Figure 8 in red.

We find the bus' velocity in unit-vector notation at $t = 2$ h as

$$\begin{aligned} \mathbf{v}(2) &= \left(-\tfrac{3}{10}\,2^2 + 4\cdot 2 - 10\right)\mathbf{i} + (-2\cdot 2 + 2)\,\mathbf{j} \\ &= (-3.20\,\mathbf{i} - 2.00\,\mathbf{j})\ \text{km/h}. \end{aligned}$$

In Figure 8 this vector is drawn in blue; notice how beautifully and flawlessly $\mathbf{v}(2)$ osculates the path of the bus! The magnitude of $\mathbf{v}(2)$ is calculated to be

$$v(2) = \sqrt{(-3.20)^2 + (-2.00)^2} \approx 3.77\ \text{km/h}$$

and the angle it makes with respect to the positive x-direction is

$$\beta = \arctan\left(\frac{-2.00}{-3.20}\right) = 32.005^\circ + 180^\circ \approx 212^\circ$$

which is also indicated in Figure 8.

What about the bus' acceleration at $t = 2$ h? We have in unit-vector notation

$$\mathbf{a}(2) = \left(-\tfrac{3}{5}\cdot 2 + 4\right)\mathbf{i} - 2\,\mathbf{j} = (2.80\,\mathbf{i} + 2.00\,\mathbf{j})\ \text{km/h}^2$$

which is shown in Figure 9 in magenta. Its magnitude is

$$a(2) = \sqrt{(2.80^2 - 2.00^2} \approx 3.44\ \text{km/h}^2$$

and the angle it makes with positive x-direction is

$$\gamma = \arctan\left(\frac{-2.00}{2.80}\right) \approx -35.5^\circ$$

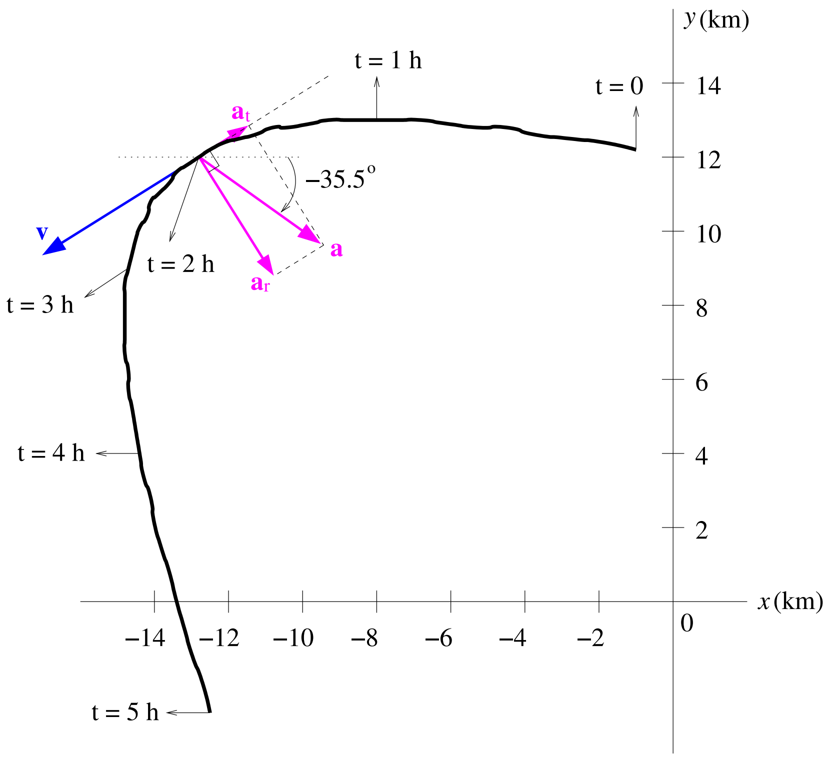

Figure 9: The velocity and acceleration vectors of the bus at $t = 2$ h.

In Figure 9 we have decomposed the acceleration vector $\mathbf{a}(2)$ into its tangential and radial components. We note that $\mathbf{v}$ and $\mathbf{a}_\mathrm{t}$ at $t = 2$ h are anti-parallel, meaning that the speed v of the bus at this moment is *decreasing* (as a prudent driver should do as negotiating a sharp bend!). We also see that the existence of a fairly large-magnitude $\mathbf{a}_\mathrm{r}$ causes the bus to change its direction sharply downward, as it is clearly illustrated in Figure 9. We can numerically verify these two observations. The best way to do this to calculate the speed v and the corresponding angle at $t = 2.1$ h:

$$\begin{aligned} \mathbf{v}(2.1) &= \left[-\tfrac{3}{10}\,(2.1)^2 + (4)(2.1) - 10\right]\mathbf{i} + \left[-(2)(2.1) + 2\right]\mathbf{j} \\ &= (-2.923\,\mathbf{i} - 2.20\,\mathbf{j})\ \text{km/h}. \end{aligned}$$

$$v(2.1) = \sqrt{(-2.923)^2 + (-2.20)^2} \approx 3.66\ \text{km/h}$$

which is less than $v(2) \approx 3.77$ km/h, as we claimed. As to the angle that $\mathbf{v}(2.1)$ makes with the positive x-direction, we have

$$\beta(2.1) = \arctan\left(\frac{-2.20}{-2.923}\right) = 36.967^\circ + 180^\circ \approx 217^\circ$$

which is significantly larger that $\beta(2) \approx 212^\circ$, indicating that the bus has turned downward by $\sim\ 5^\circ$.

Motion with Constant Acceleration: 3D Consideration

- In Chapter 2 we studied 1D motion with constant acceleration; with a slight change in notation, we repeat the equations obtained there for an object whose motion is restricted to the x-axis:

$$\begin{aligned} a_x &= \text{const}_1 \\ v_x &= v_{x0} + a_x t \\ x &= x_0 + v_{x0}t + \tfrac{1}{2}\,a_x t^2 \\ v_x^2 &= v_{x0}^2 + 2a_x\,(x - x_0) \end{aligned}$$

  Here the subscript x in a_x, for example, indicates that we are considering the acceleration in the x-direction. Similarly, v_{x0} corresponds to the initial velocity in the x-direction.

- We now consider a general 3D motion with constant acceleration:

$$\mathbf{a} = \mathbf{const}.$$

With constant acceleration we mean that all the three components of the acceleration vector are constant:

$$\mathbf{a} = \mathbf{const.} \quad \Rightarrow \quad a_x = \text{const}_1, \quad a_y = \text{const}_2, \quad a_z = \text{const}_3$$

This in turn means that both the magnitude and the direction of the acceleration vector remain constant at all times.

- It is an empirical fact that the complicated 3D motion of an object with constant acceleration can be simplified considerably by analyzing the motion for each dimension seperately. In other words, part of the motion on an axis does not have any influence on other parts on the other axes.[1] Therefore, we can write sets of equations for the parts of the motion on the y- and z-axes, just similar and in addition to that we wrote above for the part on the x-axis:

$$\begin{aligned} a_y &= \text{const}_2 & \qquad a_z &= \text{const}_3 \\ v_y &= v_{y0} + a_y t & v_z &= v_{z0} + a_z t \\ y &= y_0 + v_{y0}t + \tfrac{1}{2}\,a_y t^2 & z &= z_0 + v_{z0}t + \tfrac{1}{2}\,a_z t^2 \\ v_y^2 &= v_{y0}^2 + 2a_y\,(y - y_0) & v_z^2 &= v_{z0}^2 + 2a_z\,(z - z_0) \end{aligned}$$

- It is the power of mathematics that the first three equations[2] in the above sets can be written succinctly in vector language as

$$\begin{aligned} \mathbf{a} &= \text{const.} \\ \mathbf{v} &= \mathbf{v}_0 + \mathbf{a}t \\ \mathbf{r} &= \mathbf{r}_0 + \mathbf{v}_0 t + \tfrac{1}{2}\,\mathbf{a}t^2 \end{aligned}$$

where $\mathbf{v} = \mathbf{v}(t)$ is the velocity of the object at time t; $\mathbf{v}_0 = \mathbf{v}(0)$ is its initial velocity at $t = 0$; $\mathbf{r} = \mathbf{r}(t)$ is its position at t; and $\mathbf{r}_0 = \mathbf{r}(0)$ is its initial position at $t = 0$.

[1] This fact is actually a direct result of the so-called **superposition principle**, which we will mention again in the following chapter.

[2] The last equations in these sets, in the form $v_x^2 = v_{x0}^2 + 2a_x\,(x - x_0)$, cannot be written in pure vector form; nonetheless, this form will pave the way for an informal introduction to the concept of (kinetic) energy in Chapter 7.

Motion with Constant Acceleration in a Plane: Projectile Motion

- The word **projectile** is a generic name for any object which is *projected* (or fired or launched) from a mechanism such as a gun, weapon or catapult, flies through the air, and maybe lands some distance away.

- Three reasonable assumptions facilitate the analysis of **projectile motion**: (1) the earth is assumed to be flat over some small horizontal range of the projectile;[3] (2) air resistance is neglected;[4] and most importantly, (3) the projectile is assumed to be subject only to the gravitational force, which results in a constant[5] downward freefall acceleration of magnitude $g = 9.80$ m/s^2. Under these conditions, projectile motion takes place in a *vertical* plane, so that it is essentially a 2D motion with a *constant* acceleration.

- Throughout this chapter, we assume that the projectile motion under question will be on the vertical xy-plane, with the x-axis being the *horizontal* direction and the y-axis being the *vertical* direction.

- Since there is no horizontal acceleration, we have

  $$a_x = 0.$$

  This means that the horizontal velocity of the projectile remains always constant, being equal *at any time* to the initial velocity:

  $$v_x = v_{x0} = \text{const.}$$

- The sole vertical acceleration is always downward; it is convenient to write it as[6]

  $$a_y = -g = -9.80 \text{ m/s}^2.$$

[3] This assumption is valid if this *small* horizontal range is much less than the radius of the earth.

[4] This is the weakest assumption to validate. Most of the time we *cannot* ignore air resistance, but its inclusion makes it difficult to analyze the motion. Nevertheless, if the projectile does not fly too fast, the neglect of air resistance is a fairly legitimate approximation.

[5] With constant freefall acceleration we mean that we neglect the variation of g with altitude, which is reasonable if projectile motion takes place near the surface of the earth.

[6] It might be useful for a diligent student to note that the projectile's acceleration can also be written in the vector form as

$$\mathbf{a} = g\,(-\mathbf{j}) = -g\,\mathbf{j}.$$

- With these choises, the set equations developed in the preceding section will take the following *general* form:

$$
\begin{aligned}
a_x &= 0 & a_y &= -g \\
v_x &= v_{x0} & v_y &= v_{y0} - gt \\
x &= x_0 + v_{x0} t & y &= y_0 + v_{y0} t - \tfrac{1}{2} g t^2 \\
& & v_y^2 &= v_{y0}^2 - 2g\,(y - y_0)
\end{aligned}
$$

Example: Symmetric Projectile Motion

- We now analyze the symmetric projectile motion as an example. We mean with “symmetric” that the launching and landing points of the projectile are on the same horizontal line (surface). As in Figure 10, taking the origin of the xy-plane as the launching point will simplify a bit the redundant notation[7]:

$$x_0 = 0 \qquad \text{and} \qquad y_0 = 0.$$

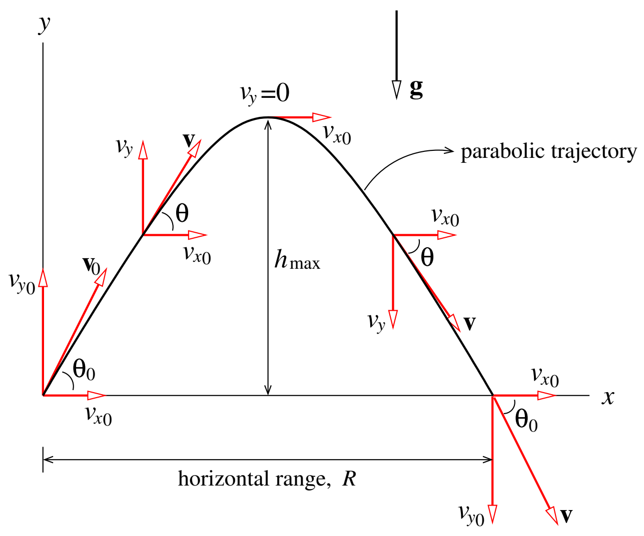

Figure 10: A symmetric projectile motion.

[7]You can always make the launching take place at the origin with a suitable change of coordinates.

- Let the projectile be launched with an *initial velocity* $\mathbf{v}_0$ which makes an *initial angle* θ_0 with the positive x-axis. We have

$$\mathbf{v}_0 = v_{x0}\mathbf{i} + v_{y0}\mathbf{j}$$

where v_{x0} and v_{y0} are the respective initial speeds in the x- and y-directions, which are found from Figure 10 as

$$v_{x0} = v_0 \cos\theta_0 \qquad \text{and} \qquad v_{y0} = v_0 \sin\theta_0$$

- Now there are only two equations that govern the motion on the horizontal x-direction. The first one is

$$v_x = v_{x0} = v_0 \cos\theta_0 = \text{const.}$$

The constancy of the horizontal velocity is clearly illustrated in Figure 10, where the speed v_{x0} has been associated everywhere with the *same* horizontal vector.

- The second equation gives the x-ccordinate of the projectile:

$$x = x_0 + v_{x0}t = 0 + (v_0 \cos\theta_0)\, t \quad \Rightarrow \quad x = (v_0 \cos\theta_0)\, t$$

There is nothing special about this result. Since v_0 and θ_0 are constant, this equation tells us that the projectile's x-coordinate increases linearly with time t.

- The projectile's vertical motion can be analyzed by three equations. The first two are for the vertical speed:

$$v_y = v_{y0} - gt \quad \Rightarrow \quad v_y = v_0 \sin\theta_0 - gt$$

Here this result tells us that the vertical speed v_y, which has an initial positive value $v_{y0} = v_0 \sin\theta_0$, decreases as time goes on with the constant accelertaion $-g$, then it becomes zero at its *highest* point, and then it reverses its direction toward the earth, that is, its numerical value becomes *negative.* The magnitude of this negative value increases with time at every moment and the projectile hits the ground (in this special symmetric problem) *with the same vertical speed as its initial vertical speed.* These are all markedly shown in Figure 10.

- The second equation relates the square of the initial vertical speed to that of the vertical speed at any later time:

$$v_y^2 = v_{y0}^2 - 2g\,(y - y_0) = (v_0 \sin\theta_0)^2 - 2g\,(y - 0)$$

$$\Rightarrow \quad v_y^2 = (v_0 \sin\theta_0)^2 - 2gy$$

- For the projectile's position in the vertical direction, we have

$$y = y_0 + v_{y0}t - \tfrac{1}{2}\,gt^2 = 0 + (v_0 \sin\theta_0)\,t - \tfrac{1}{2}\,gt^2$$

From the equation for x on the previous page, we obtain the general expression for time t as

$$x = (v_0 \cos\theta_0)\,t \quad \Rightarrow \quad t = \frac{x}{v_0 \cos\theta_0}$$

Substituting this into the equation for y above, we get

$$y = (v_0 \sin\theta_0) \cdot \frac{x}{v_0 \cos\theta_0} - \tfrac{1}{2}\,g\left(\frac{x}{v_0 \cos\theta_0}\right)^2$$

whence

$$y = (\tan\theta_0)\,x - \left(\frac{g}{2v_0^2 \cos^2\theta_0}\right)x^2$$

Here the expressions inside the parantheses are all constant so that this equation is of the form $y = ax^2 + bx$, which is an equation of *parabola* passing through the origin. In other words, the path of a projectile is a *parabolic* ***trajectory***.

- At the instant the projectile's vertical speed v_y is zero, it will be in its maximum height h_{max}, as shown in Figure 10. So equating the equation for v_y to zero, we find the time t_{max} required for the particle to reach its maximum height:

$$v_y = 0 = v_0 \sin\theta_0 - gt_{\text{max}} \quad \Rightarrow \quad t_{\text{max}} = \frac{v_0 \sin\theta_0}{g}$$

Plugging this result into the equation for y, we obtain

$$\begin{aligned} y = h_{\text{max}} &= (v_0 \sin\theta_0)\,t_{\text{max}} - \tfrac{1}{2}\,gt_{\text{max}}^2 \\ &= (v_0 \sin\theta_0) \cdot \frac{v_0 \sin\theta_0}{g} - \tfrac{1}{2}\,g \cdot \left(\frac{v_0 \sin\theta_0}{g}\right)^2 \end{aligned}$$

which leads to the maximum height for this symmetric projectile motion:

$$h_{\text{max}} = \frac{v_0^2 \sin^2\theta_0}{2g}$$

- In Figure 10 is also shown the **horizontal range** R of the projectile which is the horizontal distance between its launching point (i.e., the origin) and its landing point whose coordinates are $(x, y) = (R, 0)$. We see that the time t_R required for the particle to reach this point is twice the time $t_{\max}$ required to reach its maximum height, which we found above:

  $$t_R = 2t_{\max} = \frac{2v_0 \sin\theta_0}{g}$$

  Insertion t_R into the equation for x results is the particle's horizontal range as

  $$x = R = (v_0 \cos\theta_0)\, t_R = (v_0 \cos\theta_0) \cdot \frac{2v_0 \sin\theta_0}{g} = \frac{2v_0^2 \sin\theta_0 \cos\theta_0}{g}$$

  Since

  $$2 \sin\theta_0 \cos\theta_0 = \sin 2\theta_0$$

  we write the expression for R in a more concise form as

  $$R = \frac{v_0^2}{g} \sin 2\theta_0$$

  Don't forget that this result is valid if and only if we have a *symmetric* projectile motion; if the launching and landing points are not on the same horizontal level, you cannot use this result.

- The expression for R above has two important consequences. The first one is that R is maximum when $\sin 2\theta_0 = 1$, which amounts to $2\theta_0 = 90°$ or $\theta_0 = 45°$.

- The second consequence follows from the expression

  $$\sin 2\theta_0 = 2 \sin\theta_0 \cos\theta_0$$

  which tells us that for two **complementary** initial angles (i.e., angles which add up to $90°$), say $30°$ and $60°$, we have the same $\sin 2\theta_0$ values, because $\sin 30° = \cos 60°$, and vice versa. We then draw the conclusion that two complementary initial angles give rise to the same horizontal range R for a symmetric projectile motion, as Figure 11 clearly illustrates for $\theta_0 = 30°$ and $60°$.

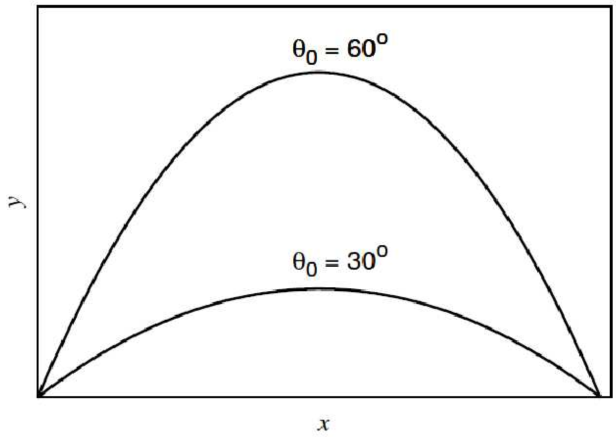

Figure 11: Two complementary initial angles lead to the same horizontal range R for a symmetric projectile motion.

Uniform Circular Motion

- As the one in Figure 12, an object orbiting a fixed point at *constant* (or, *uniform*) *speed* is said to be in **uniform circular motion**.

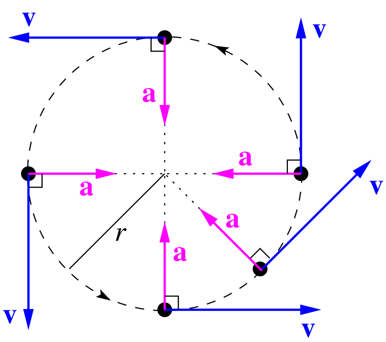

Figure 12: An object undergoing a uniform circular motion.

- Although its speed v is constant, the velocity $\mathbf{v}$ of the object *does* change, because its direction changes continuously, as is seen in Figure 12. Since acceleration by definition is the change in velocity $\mathbf{v}$ in time, the object *does* continuously undergo an acceleration.

- Out of its importance, it is convenient to state first the underlying properties of uniform circular motion:

    1. Uniform circular motion takes place in a plane, so that it is a 2D motion.
    2. The path of the object might also be a circular arc, *not* necessarily a complete circle.
    3. The speed v of the object is always constant around its circular path.
    4. The velocity $\mathbf{v}$ of the object is everywhere tangent to its circular path (Figure 12).
    5. The acceleration $\mathbf{a}$ of the object is everywhere perpendicular to the velocity vector and is always directed to the center of the circular path. In other words, $\mathbf{a}$ is *radially inward* (Figure 12). Because of these, the object's acceleration $\mathbf{a} = \mathbf{a}_{\mathrm{r}}$ is called **radial acceleration**, or frequently **centripetal acceleration**.[8]

        We shall show shortly that the magnitude of this radial acceleration is given by

        $$\boxed{a = a_{\mathrm{r}} = \frac{v^2}{r}}$$

        where v is the speed of the object, and r is the radius of its circular path. You can easily verify that this expression has the correct unit of acceleration, $\mathrm{m/s^2}$.
    6. If the object's path is a complete circle, its acceleration vector has *no* component along its velocity vector; that is, there is no tangential acceleration vector: $\mathbf{a}_{\mathrm{t}} = \mathbf{0}$.
    7. We call the time T required for the object to complete one revolution as the **period** of the motion. Since the object travels a distance of $2\pi r$ in one revolution, it follows from the definition of speed that $v = 2\pi r/T$, or

        $$\boxed{T = \frac{2\pi r}{v} \qquad \text{(s)}}$$

[8] The word "centripetal" comes from Latin *centrum*, meaning "center," and *petere*, meaning "to seek," so that centripetal literally means "center seeking."

- Let's now prove that acceleration $\mathbf{a}$ is directed toward the center of the circle and that its magnitude is given by $a = a_r = v^2/r$. To do so we will use Figure 13, where the object is at position 1 at time t_1 and at position 2 to at a later time t_2. The object's velocity is $\mathbf{v}_1$ at 1 and $\mathbf{v}_2$ at 2, with $v_1 = v_2 = v$; that is, the object's velocity changes only its direction while its magnitude remains constant.

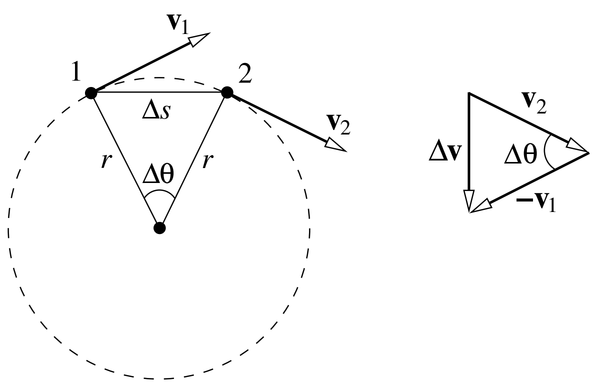

Figure 13: Figures for the derivation of $a = a_r = v^2/r$.

The *average* acceleration of the object when it goes from position 1 to position 2 is

$$\mathbf{a}_{\text{avg}} = \frac{\Delta\mathbf{v}}{\Delta t} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{t_2 - t_1}$$

We are then required to determine $\Delta\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$, which we have performed graphically in the vector triangle, shown also Figure 13. We notice immediately that $\Delta\mathbf{v}$ is directed toward the center of the circle! This conclusion is also valid if we let time Δt be infinitesimally small.

To determine the magnitude of $\mathbf{a}_{\text{avg}}$ when $\Delta t \to 0$, we write the magnitude of the first part of the preceding equation as

$$a_{\text{avg}} = \frac{\Delta v}{\Delta t}$$

Figure 13 shows two *congruent* (or *similar*) triangles, one on the left with sides Δs and r and one on the right with sides Δv and v. The congruency condition gives us

$$\frac{\Delta s}{r} = \frac{\Delta v}{v} \qquad \text{or} \qquad \Delta v = \frac{v}{r}\,\Delta s$$

With this result we have

$$a_{\text{avg}} = \frac{\Delta v}{\Delta t} = \frac{v\,\Delta s/r}{\Delta t} = \frac{v}{r}\,\frac{\Delta s}{\Delta t}$$

We then take the limit of this expression as $\Delta t \to 0$; this means that the distance between positions 1 and 2 is infinitesimally small. In this limit the ratio $\Delta s/\Delta t$ approaches the speed v of the object and the average velocity $\mathbf{a}_{\text{avg}}$ appraches the *instantaneous* acceleration a, which is the centripetal acceleration a_{r} of the object,

$$\lim_{\Delta t \to 0} a_{\text{avg}} = a = a_{\text{r}} = \lim_{\Delta t \to 0} \frac{v}{r} \frac{\Delta s}{\Delta t} = \frac{v}{r} \cdot v = \frac{v^2}{r}$$

which is what we set out to prove.

Key Words, Phrases, and Equations

- Position vector:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \qquad \text{(m)}$$

$$r \equiv |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} \qquad \text{(m)}$$

- Displacement vector:

$$\begin{aligned} \Delta \mathbf{r} &= \mathbf{r}_2 - \mathbf{r}_1 \\ &= (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k} \qquad \text{(m)} \\ &= \Delta x\,\mathbf{i} + \Delta y\,\mathbf{j} + \Delta z\,\mathbf{k} \end{aligned}$$

- Average velocity:

$$\mathbf{v}_{\text{avg}} = \frac{\Delta \mathbf{r}}{\Delta t} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{t_2 - t_1} \qquad \text{(m/s)}$$

- (Instantaneous) velocity:

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt} \qquad \text{(m/s)}$$

$$\begin{aligned} \mathbf{v} &= \mathbf{v}_x + \mathbf{v}_y + \mathbf{v}_z \\ &= v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k} \qquad \text{(m/s)} \\ &= \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k} \end{aligned}$$

- Speed:

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} \qquad (\mathrm{m/s})$$

- Average acceleration:

$$\mathbf{a}_{\rm avg} = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{t_2 - t_1} \qquad \left(\mathrm{m/s^2}\right)$$

- (Instantaneous) acceleration:

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt} \qquad \left(\mathrm{m/s^2}\right)$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \qquad \left(\mathrm{m/s^2}\right)$$

$$\begin{aligned} \mathbf{a} &= \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z \\ &= a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k} \\ &= \frac{dv_x}{dt}\,\mathbf{i} + \frac{dv_y}{dt}\,\mathbf{j} + \frac{dv_z}{dt}\,\mathbf{k} \qquad \left(\mathrm{m/s^2}\right) \\ &= \frac{dx^2}{dt^2}\,\mathbf{i} + \frac{dy^2}{dt^2}\,\mathbf{j} + \frac{dz^2}{dt^2}\,\mathbf{k} \end{aligned}$$

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2} \qquad \left(\mathrm{m/s^2}\right)$$

- Tangential and radial accelerations:

$$a_{\rm t} = \frac{dv}{dt} \qquad \text{and} \qquad a_{\rm r} = \frac{v^2}{r}$$

- Motion with constant acceleration:

$$\begin{aligned} \mathbf{a} &= \text{const.} \\ \mathbf{v} &= \mathbf{v}_0 + \mathbf{a}t \\ \mathbf{r} &= \mathbf{r}_0 + \mathbf{v}_0 t + \tfrac{1}{2}\,\mathbf{a}t^2 \end{aligned}$$

- Projectile motion:

$$
\begin{aligned}
a_x &= 0 & a_y &= -g \\
v_x &= v_{x0} & v_y &= v_{y0} - gt \\
x &= x_0 + v_{x0}t & y &= y_0 + v_{y0}t - \tfrac{1}{2}\,gt^2 \\
& & v_y^2 &= v_{y0}^2 - 2g\,(y - y_0)
\end{aligned}
$$

- Parabolic trajectory, maximum altitude h_{max}, horizontal range R, complementary angles.

- Radial (centripetal) acceleration in a uniform circular motion:

$$a = a_{\mathrm{r}} = \frac{v^2}{r}$$

- Period of a uniform circular motion:

$$T = \frac{2\pi r}{v} \qquad \text{(s)}$$