INDUSTRIAL AUTOMATION & ROBOTICS TECHNOLOGY

Dynamic Analysis and Forces

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Materials used

- Chapter 4, Introduction to Robotics, Saeed B. Niku

EFFECTIVE MOMENTS OF INERTIA

- To Simplify the equation of motion, Equations can be rewritten in symbolic form.

$$\begin{bmatrix} T_i \\ T_j \end{bmatrix} = \begin{bmatrix} D_{ii} & D_{ij} \\ D_{ji} & D_{jj} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_i \\ \ddot{\theta}_j \end{bmatrix} + \begin{bmatrix} D_{iii} & D_{ijj} \\ D_{jii} & D_{jjj} \end{bmatrix} \begin{bmatrix} \dot{\theta}_i^2 \\ \dot{\theta}_j^2 \end{bmatrix} + \begin{bmatrix} D_{iij} & D_{iji} \\ D_{jij} & D_{jji} \end{bmatrix} \begin{bmatrix} \dot{\theta}_i \dot{\theta}_j \\ \dot{\theta}_j \dot{\theta}_i \end{bmatrix} + \begin{bmatrix} D_i \\ D_j \end{bmatrix}$$

Inertia

Centripetal forces

Coriolis acceleration

gravity

D_{ii} is known as effective inertia
D_{ij} is known as coupling inertia between joint i and j

DYNAMIC EQUATIONS FOR MULTIPLE-DEGREE-OF-FREEDOM ROBOTS

Kinetic Energy

- Equations for a multiple-degree-of-freedom robot are very long and complicated, but can be found by calculating the kinetic and potential energies of the links and the joints, by defining the Lagrangian and by differentiating the Lagrangian equation with respect to the joint variables.

- **The kinetic energy of a rigid body with motion in three dimension :**

$$K = \frac{1}{2} m \overline{V}^2 + \frac{1}{2} \overline{\omega}\, \overline{h}_G$$

- **The kinetic energy of a rigid body in planar motion**

$$K = \frac{1}{2} m \overline{V}^2 + \frac{1}{2} \overline{I} \omega^2$$

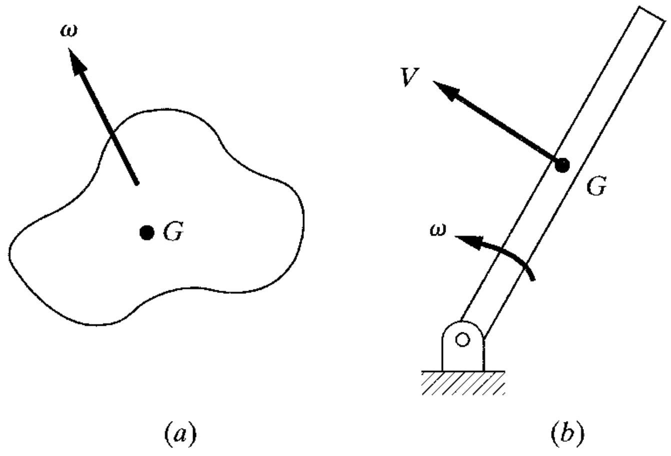
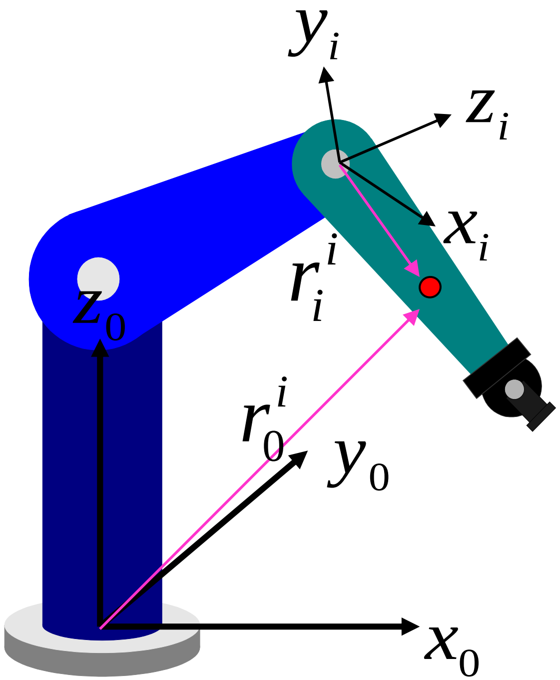


Fig. 4.7 A rigid body in three-dimensional motion and in plane motion.

Velocity of a link

$$r_i^i = \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix}$$ A point fixed in link *i* and expressed w.r.t. the *i*-th frame



Same point w.r.t the base frame

$$r_0^i = T_0^i r_i^i = (T_0^1 T_1^2 \cdots T_{i-1}^i) r_i^i$$

Velocity of a link

Velocity of point r_i^i expressed w.r.t. *i*-th frame is zero

$$\dot{r}_i^i = 0$$

Velocity of point r_i^i expressed w.r.t. base frame is:

$$V_i \equiv V_0^i = \frac{d}{dt} r_0^i = \frac{d}{dt}(T_0^1 T_1^2 \cdots T_{i-1}^i) r_i^i$$

$$= \dot{T}_0^1 T_1^2 \cdots T_{i-1}^i r_i^i + T_0^1 \dot{T}_1^2 \cdots T_{i-1}^i r_i^i +$$

$$\cdots + T_0^1 T_1^2 \cdots \dot{T}_{i-1}^i r_i^i + T_0^i \dot{r}_i^i = \left(\sum_{j=1}^{i} \frac{\partial T_0^i}{\partial q_j} \dot{q}_j\right) r_i^i$$

Velocity of a link

- Rotary joints, $q_i = \theta_i$

$$T_{i-1}^{i} = \begin{bmatrix} C\theta_i & -C\alpha_i S\theta_i & S\alpha_i S\theta_i & a_i C\theta_i \\ S\theta_i & C\alpha_i C\theta_i & -S\alpha_i C\theta_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\frac{\partial T_{i-1}^{i}}{\partial q_i} = \begin{bmatrix} -S\theta_i & -C\alpha_i C\theta_i & S\alpha_i C\theta_i & -a_i S\theta_i \\ C\theta_i & -C\alpha_i S\theta_i & S\alpha_i S\theta_i & a_i C\theta_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Q_i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial T_{i-1}^{i}}{\partial q_i} = Q_i T_{i-1}^{i} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C\theta_i & -C\alpha_i S\theta_i & S\alpha_i S\theta_i & a_i C\theta_i \\ S\theta_i & C\alpha_i C\theta_i & -S\alpha_i C\theta_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Velocity of a link

- Prismatic joint, $q_i = d_i$

$$T_{i-1}^{i} = \begin{bmatrix} C\theta_i & -C\alpha_i S\theta_i & S\alpha_i S\theta_i & a_i C\theta_i \\ S\theta_i & C\alpha_i C\theta_i & -S\alpha_i C\theta_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Q_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\frac{\partial T_{i-1}^{i}}{\partial q_i} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial T_{i-1}^{i}}{\partial q_i} = Q_i T_{i-1}^{i}$$

Velocity of a Link

The effect of the motion of joint j on all the points on link i

$$U_{ij} \equiv \frac{\partial T_0^i}{\partial q_j} = \begin{cases} T_0^{j-1} Q_j T_{j-1}^i & for \quad j \le i \\ 0 & for \quad j > i \end{cases}$$

The interaction effects of the motion of joint j and joint k on all the points on link i

$$\frac{\partial U_{ij}}{\partial q_k} \equiv U_{ijk} = \begin{cases} T_0^{j-1} Q_j T_{j-1}^{k-1} Q_k T_{k-1}^i & i \ge k \ge j \\ T_0^{k-1} Q_k T_{k-1}^{j-1} Q_j T_{j-1}^i & i \ge j \ge k \\ 0 & i < j \quad or \quad i < k \end{cases}$$

Velocity of a link

- **The velocity of a point along a robot's link can be defined by differentiating the position equation of the point.**

$$p_i = {}^{R}T_i r_i = {}^{0}T_i r_i$$

$$V_i \equiv V_0^i = \frac{d}{dt} r_0^i = \frac{d}{dt}(T_0^1 T_1^2 \cdots T_{i-1}^i) r_i^i = \left(\sum_{j=1}^{i} \frac{\partial T_0^i}{\partial q_j} \dot{q}_j\right) r_i^i = \left(\sum_{j=1}^{i} U_{ij} \dot{q}_j\right) r_i^i$$

Kinetic energy of link i

- Kinetic energy of a particle with differential mass dm in link i

$$dK_i = \frac{1}{2}(\dot{x}_i^{\ 2} + \dot{y}_i^{\ 2} + \dot{z}_i^{\ 2})dm = \frac{1}{2} trace(V_i V_i^{\ T})dm$$

$$= \frac{1}{2} Tr\left[\sum_{p=1}^{i} U_{ip} \dot{q}_p r_i^i \left(\sum_{r=1}^{i} U_{ir} \dot{q}_r r_i^i\right)^T \right] dm$$

$$= \frac{1}{2} Tr\left[\sum_{p=1}^{i} \sum_{r=1}^{i} U_{ip} r_i^i r_i^{iT} U_{ir}^T \dot{q}_p \dot{q}_r \right] dm$$

$$= \frac{1}{2} Tr\left[\sum_{p=1}^{i} \sum_{r=1}^{i} U_{ip} (r_i^i dm r_i^{iT}) U_{ir}^T \dot{q}_p \dot{q}_r \right]$$

$$Tr(A) \equiv \sum_{i=1}^{n} a_{ii}$$

Kinetic energy of link i

$$K_i = \int dK_i = \frac{1}{2} Tr\left[\sum_{p=1}^{i} \sum_{r=1}^{i} U_{ip} \left(\int r_i^i r_i^{iT} dm \right) U_{ir}^T \dot{q}_p \dot{q}_r \right]$$

$$I_i = \int r_i^i r_i^{iT} dm = \begin{bmatrix} \int x_i^2 dm & \int x_i y_i dm & \int x_i z_i dm & \int x_i dm \\ \int x_i y_i dm & \int y_i^2 dm & \int y_i z_i dm & \int y_i dm \\ \int x_i z_i dm & \int y_i z_i dm & \int z_i^2 dm & \int z_i dm \\ \int x_i dm & \int y_i dm & \int z_i dm & \int dm \end{bmatrix}$$

$$\bar{r}_i^i = \begin{bmatrix} \bar{x}_i \\ \bar{y}_i \\ \bar{z}_i \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-I_{xx} + I_{yy} + I_{zz}}{2} & I_{xy} & I_{xz} & m_i \bar{x}_i \\ I_{xy} & \frac{I_{xx} - I_{yy} + I_{zz}}{2} & I_{yz} & m_i \bar{y}_i \\ I_{xz} & I_{yz} & \frac{I_{xx} + I_{yy} - I_{zz}}{2} & m_i \bar{z}_i \\ m_i \bar{x}_i & m_i \bar{y}_i & m_i \bar{z}_i & m_i \end{bmatrix}$$

$$\bar{x}_i = \frac{1}{m_i} \int x_i dm$$

Center of mass

Pseudo-inertia matrix of link i

Manipulator Dynamics

- Total kinetic energy of a robot arm

$$K = \sum_{i=1}^{n} K_i = \frac{1}{2} \sum_{i=1}^{n} Tr\left[\sum_{p=1}^{i} \sum_{r=1}^{i} U_{ip} \left(\int r_i^i r_i^{iT} dm \right) U_{ir}^T \dot{q}_p \dot{q}_r \right]$$

$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{p=1}^{i} \sum_{r=1}^{i} \left[Tr(U_{ip} I_i U_{ir}^T) \dot{q}_p \dot{q}_r \right] + \frac{1}{2} \sum_{i=1}^{n} I_{i(act)} \dot{q}_i^2$$

Scalar quantity, function of q_i and $\dot{q}_i$, $i = 1, 2, \cdots n$

I_i : Pseudo-inertia matrix of link i, dependent on the mass distribution of link i and are expressed w.r.t. the i-th frame,

Need to be computed once for evaluating the kinetic energy

Manipulator Dynamics

- Potential energy of link i

$\bar{r}_0^{\,i}$: Center of mass w.r.t. base frame

$$P_i = -m_i g \bar{r}_0^{\,i} = -m_i g (T_0^i \bar{r}_i^{\,i})$$

$\bar{r}_i^{\,i}$: Center of mass w.r.t. i-th frame

$$g = (g_x, g_y, g_z, 0)$$

$$|g| = 9.8 m / \sec^2$$

g : gravity row vector expressed in base frame

- Potential energy of a robot arm

$$P = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} [-m_i g^T (T_0^i \bar{r}_i^{\,i})]$$

Function of q_i

Manipulator Dynamics

- Lagrangian function

$$L = K - P = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i}\left[Tr(U_{ij}I_iU_{ik}{}^{T})\dot{q}_j\dot{q}_k\right] + \frac{1}{2}\sum_{i=1}^{n}I_{i(act)}\dot{q}_i^2$$

$$+\sum_{i=1}^{n}m_i g(T_0^i\overline{r}_i^{\,i})$$

DYNAMIC EQUATIONS FOR MULTIPLE-DEGREE-OF-FREEDOM ROBOTS

Robot's Equations of Motion

- **The Lagrangian is differentiated to form the dynamic equations of motion.**
- **The final equations of motion for a general multi-axis robot is below.**

$$T_i = \sum_{j=1}^{n} D_{ij}\ddot{q}_j + I_{i(act)}\ddot{q}_i + \sum_{j=1}^{n}\sum_{k=1}^{n} D_{ijk}\dot{q}_j\dot{q}_k + D_i$$

where, $D_{ij} = \sum_{p=\max(i,j)}^{n} Trace(U_{pj} J_p U_{pi}^T)$

$$D_{ijk} = \sum_{p=\max(i,j,k)}^{n} Trace(U_{pjk} J_p U_{pi}^T)$$

$$D_i = \sum_{p=i}^{n} - m_p g^T U_{pi} \bar{r}_p$$

Example 4.7 Using the aforementioned equations, derive the equations of motion for the two-degree of freedom robot arm. The two links are assumed to be of equal length.

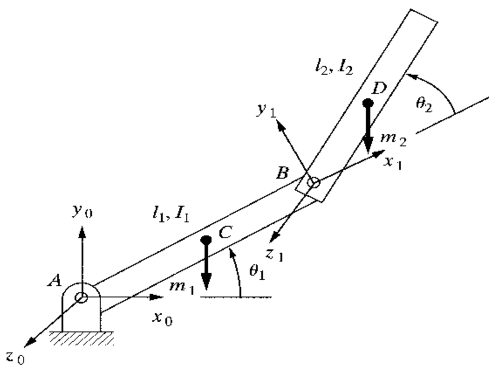


Fig. 4.8 The two-degree-of-freedom robot arm of Example 4.4

Follow the same steps as before…….

- **Write the A matrices for the two links;**
- **Develop the D_{ij}, D_{ijk} and D_i for the robot.**

Solution

- **The final equations of motion without the actuator inertia terms are the same as below.**

$$T_1=\left(\frac{1}{3}m_1l^2+\frac{4}{3}m_2l^2+m_2l^2C_2\right)\ddot{\theta}_1+\left(\frac{1}{3}m_2l^2+\frac{1}{2}m_2l^2C_2\right)\ddot{\theta}_2$$

$$+\left(\frac{1}{2}m_2l^2S_2\right)\dot{\theta}_2^2+\left(m_2l^2S_2\right)\dot{\theta}_1\dot{\theta}_2+\frac{1}{2}m_1glC_1+\frac{1}{2}m_2glC_{12}+m_2glC_1+I_{1(act)}\ddot{\theta}_1$$

$$T_2=\left(\frac{1}{3}m_2l^2+\frac{1}{2}m_2l^2C_2\right)\ddot{\theta}_1+\left(\frac{1}{3}m_2l^2\right)\ddot{\theta}_2+\left(\frac{1}{2}m_2l^2S_2\right)+\frac{1}{2}m_2glC_{12}+I_{2(act)}\ddot{\theta}_1$$

Manipulator Dynamics

- Dynamics Model

$$\tau_i = \sum_{k=1}^{n} D_{ik}\ddot{q}_k + \sum_{k=1}^{n}\sum_{m=1}^{n} h_{ikm}\dot{q}_k\dot{q}_m + C_i$$

$$D_{ik} = \sum_{j=\max(i,k)}^{n} Tr(U_{jk} I_j U_{ji}{}^T)$$

$$h_{ikm} = \sum_{j=\max(i,k,m)}^{n} Tr(U_{jkm} I_j U_{ji}{}^T)$$

$$C_i = -\sum_{j=i}^{n} m_j g U_{ji} \bar{r}_j{}^j$$

Manipulator Dynamics

- Dynamics Model of n-link Arm

$$\tau = D(q)\ddot{q} + h(q,\dot{q}) + C(q)$$

$$D = \begin{bmatrix} D_{11} & \cdots & D_{1n} \\ & \vdots & \\ D_{n1} & \cdots & D_{nn} \end{bmatrix}$$

The Acceleration-related Inertia matrix term, Symmetric

$$h(q,\dot{q}) = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$$

The Coriolis and Centrifugal terms

$$C(q) = \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix}$$

The Gravity terms

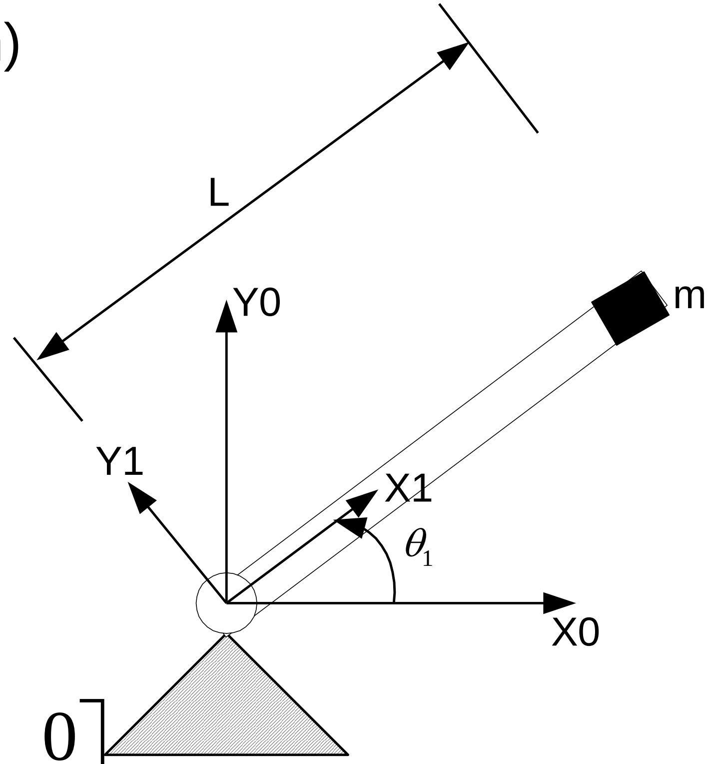
$$\tau = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix}$$

Driving torque applied on each link

Example

Example: One joint arm with point mass (m) concentrated at the end of the arm, link length is l , find the dynamic model of the robot using L-E method.

Set up coordinate frame as in the figure



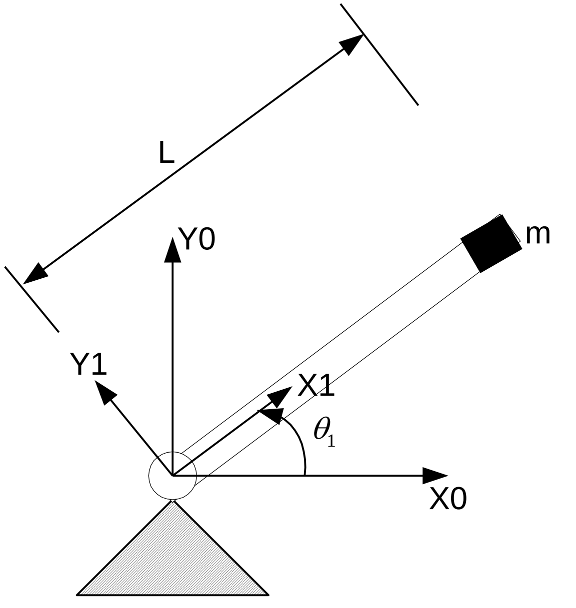
$$r_1^1 = \begin{bmatrix} l \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$g = [0, -9.8, 0, 0]$$

$$r_0^1 = T_0^1 r_1^1 = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & 0 \\ S\theta_1 & C\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} r_1^1$$

Example

$$r_0^1 = T_0^1 r_1^1 = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & 0 \\ S\theta_1 & C\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} r_1^1$$



$$V_1 = \dot{r}_0^1 = \frac{d}{dt} T_0^1 r_1^1 = Q_1 T_0^1 r_1^1 = \begin{bmatrix} -l \cdot S\theta_1 \\ l \cdot C\theta_1 \\ 0 \\ 0 \end{bmatrix} \dot{\theta}$$

Example

Kinetic energy

$$dK = \frac{1}{2} Tr(V_1 V_1^{\,T}) dm$$

$$K = \frac{1}{2}\int Tr(\begin{bmatrix} -l \cdot S\theta_1 \\ l \cdot C\theta_1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -l \cdot S\theta_1 & l \cdot C\theta_1 & 0 & 0 \end{bmatrix}) \dot{\theta}_1^{\,2} dm$$

$$= \frac{1}{2} Tr(\begin{bmatrix} l^2 \cdot (S\theta_1)^2 & -l^2 \cdot S\theta_1 \cdot C\theta_1 & 0 & 0 \\ -l^2 \cdot S\theta_1 \cdot C\theta_1 & l^2 \cdot (C\theta_1)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}) \dot{\theta}_1^{\,2} m$$

$$= \frac{1}{2}[l^2 (S\theta_1)^2 + l^2 (C\theta_1)^2] m \dot{\theta}^2 = \frac{1}{2} l^2 m \dot{\theta}_1^{\,2}$$

Example

- Potential energy

$$P = -mg(T_0^1 \bar{r}_1) = -m\begin{bmatrix}0 & -9.8 & 0 & 0\end{bmatrix}\begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & 0 \\ S\theta_1 & C\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} l \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= 9.8m \cdot l \cdot S\theta_1$$

- Lagrange function

$$L = K - P = \frac{1}{2} l^2 m \dot{\theta}_1^{\,2} - 9.8m \cdot l \cdot S\theta_1$$

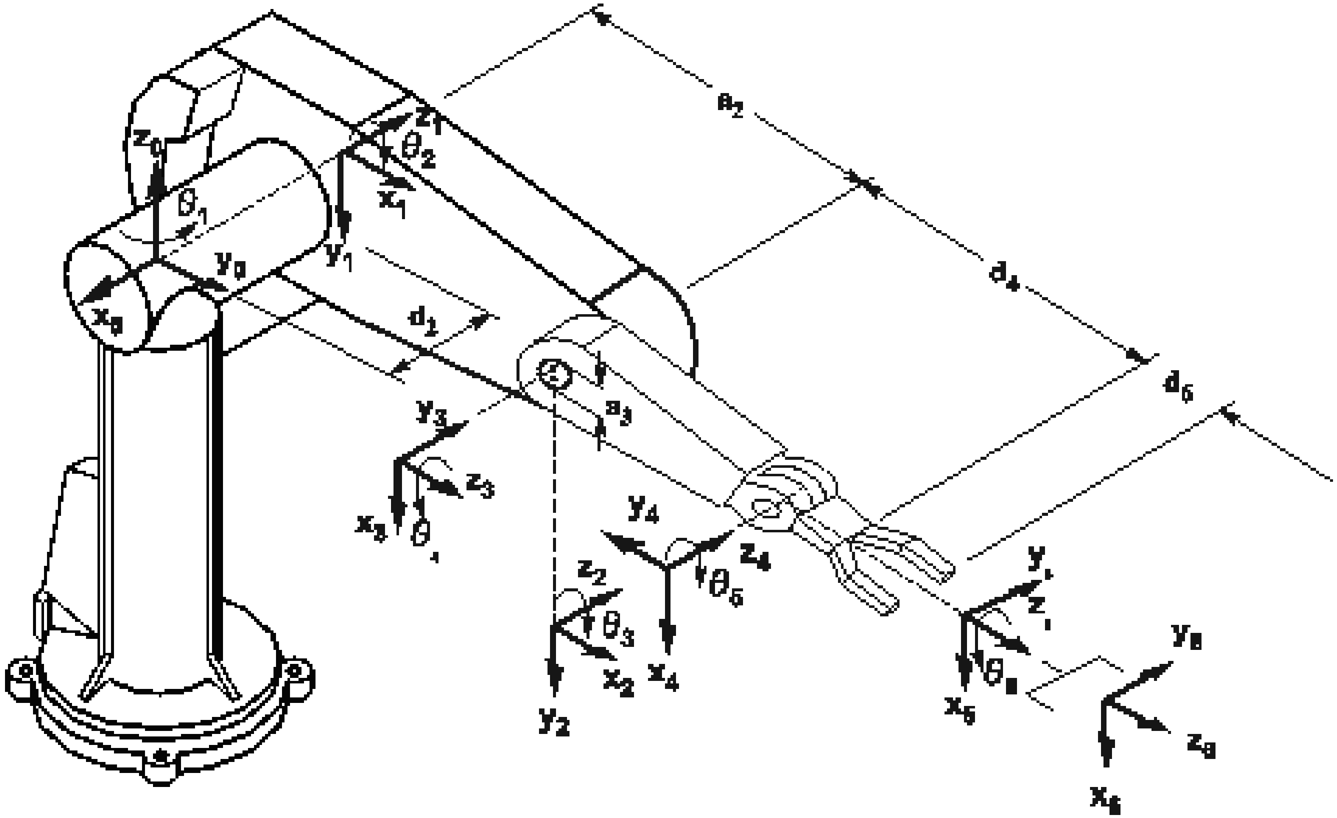
- Equation of Motion

$$\tau_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1}$$

$$= \frac{d}{dt}(l^2 m \dot{\theta}_1) - 9.8m \cdot l \cdot C\theta_1 = l^2 m \ddot{\theta}_1 - 9.8m \cdot l \cdot C\theta_1$$

Example: **Puma 560**

- Derive dynamic equations for the first 4 links of PUMA 560 robot



Example: **Puma 560**

- Set up D-H Coordinate frame
- Get robot link parameters

Joint i	θ_i	α_i	$a_i(mm)$	$d_i(mm)$
1	θ_1	-90	0	0
2	θ_2	0	431.8	-149.09
3	θ_3	90	-20.32	0
4	θ_4	-90	0	433.07
5	θ_5	90	0	0
6	θ_6	0	0	56.25

- Get transformation matrices T_{i-1}^{i}
- Get D, H, C terms

Example: **Puma 560**

- Get D, H, C terms

$$D_{ik} = \sum_{j=\max(i,k)}^{n} Tr(U_{jk} I_j U_{ji}{}^T) \qquad n = 3; i = 1,2,3$$

$$D_{11} = Tr(U_{11} I_1 U_{11}{}^T) + Tr(U_{21} I_2 U_{21}{}^T) + Tr(U_{31} I_3 U_{31}{}^T)$$

$$D_{12} = D_{21} = Tr(U_{22} I_2 U_{21}{}^T) + Tr(U_{32} I_3 U_{31}{}^T)$$

$$D_{13} = D_{31} = Tr(U_{33} I_3 U_{31}{}^T)$$

$$D_{22} = Tr(U_{22} I_2 U_{22}{}^T) + Tr(U_{32} I_3 U_{32}{}^T)$$

$$D_{23} = D_{32} = Tr(U_{33} I_3 U_{32}{}^T)$$

$$D_{33} = Tr(U_{33} I_3 U_{33}{}^T)$$

Example: **Puma 560**

- Get D, H, C terms

$$h(q,\dot{q}) = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$$

$$h_i = \sum_{k=1}^{n}\sum_{m=1}^{n} h_{ikm}\dot{q}_k\dot{q}_m$$

$$h_{ikm} = \sum_{j=\max(i,k,m)}^{n} Tr(U_{jkm} I_j U_{ji}{}^T)$$

$$h_1 = h_{111}\dot{q}_1{}^2 + h_{112}\dot{q}_1\dot{q}_2 + h_{113}\dot{q}_1\dot{q}_3 + h_{121}\dot{q}_1\dot{q}_2 + h_{122}\dot{q}_2{}^2$$
$$+ h_{123}\dot{q}_2\dot{q}_3 + h_{131}\dot{q}_3\dot{q}_1 + h_{132}\dot{q}_3\dot{q}_2 + h_{133}\dot{q}_3{}^2$$

$$h_{111} = Tr(U_{111} I_1 U_{11}{}^T) + Tr(U_{211} I_2 U_{21}{}^T) + Tr(U_{311} I_3 U_{31}{}^T)$$
$$h_{112} = Tr(U_{212} I_2 U_{21}{}^T) + Tr(U_{312} I_3 U_{31}{}^T)$$
$$h_{121} = Tr(U_{221} I_2 U_{21}{}^T) + Tr(U_{321} I_3 U_{31}{}^T) \qquad h_{113} = Tr(U_{313} I_3 U_{31}{}^T)$$
$$h_{122} = Tr(U_{222} I_2 U_{21}{}^T) + Tr(U_{322} I_3 U_{31}{}^T) \qquad h_{123} = Tr(U_{323} I_3 U_{31}{}^T)$$

....

Example: **Puma 560**

- Get D, H, C terms

$$\frac{\partial U_{ij}}{\partial q_k} \equiv U_{ijk} = \begin{cases} T_0^{j-1} Q_j T_{j-1}^{k-1} Q_k T_{k-1}^{i} & i \ge k \ge j \\ T_0^{k-1} Q_k T_{j-1}^{k-1} Q_j T_{j-1}^{i} & i \ge j \ge k \\ 0 & i < j \quad or \quad i < k \end{cases}$$

$$U_{111} = (Q_1)^2 T_0^1 \qquad U_{211} = (Q_1)^2 T_0^2 \qquad U_{311} = (Q_1)^2 T_0^3$$

$$U_{212} = U_{221} = Q_1 T_0^1 Q_2 T_1^2 \qquad U_{312} = U_{321} = Q_1 T_0^1 Q_2 T_1^3$$

$$U_{313} = Q_1 T_0^2 Q_3 T_2^3 \qquad U_{222} = T_0^1 (Q_2)^2 T_1^2 \qquad U_{322} = T_0^1 (Q_2)^2 T_1^3$$

$$U_{323} = U_{332} = T_0^1 Q_2 T_1^2 Q_3 T_2^3 \qquad U_{331} = Q_1 T_0^2 Q_3 T_2^3 \qquad U_{333} = T_0^2 Q_3 Q_2 T_2^3$$

Example: **Puma 560**

- Get D, H, C terms

$$C_i = -\sum_{j=i}^{n} m_j g U_{ji} \bar{r}_j^{\,j}$$

$$C_1 = -m_1 g U_{11} \bar{r}_1^{1} - m_2 g U_{21} \bar{r}_2^{2} - m_3 g U_{31} \bar{r}_3^{3}$$

$$C_2 = -m_2 g U_{22} \bar{r}_2^{2} - m_3 g U_{32} \bar{r}_3^{3}$$

$$C_3 = -m_3 g U_{33} \bar{r}_3^{3}$$

STATIC FORCE ANALYSIS OF ROBOTS

- **Robot Control means <u>Position Control</u> and <u>Force Control</u>.**
- **Position Control: The robot follows a prescribed path without any reactive force.**
- **Force Control: The robot encounters with unknown surfaces and manages to handle the task by adjusting the uniform depth while getting the reactive force.**

Ex) Tapping a Hole - move the joints and rotate them at particular rates to create the desired forces and moments at the hand frame.

Ex) Peg Insertion – avoid the jamming while guiding the peg into the hole and inserting it to the desired depth.

STATIC FORCE ANALYSIS OF ROBOTS

- **To Relate the joint forces and torques to forces and moments generated at the hand frame of the robot.**

$$\left[{}^{H}F\right]=\left[{}^{H}f_x \quad {}^{H}f_y \quad {}^{H}f_z \quad {}^{H}m_x \quad {}^{H}m_y \quad {}^{H}m_z\right]^T$$

- **f is the force and m is the moment along the axes of the hand frame.**

$$\delta W=\left[{}^{H}F\right]^T\left[{}^{H}D\right]=\left[T\right]^T\left[D_\theta\right]$$

- **The total virtual work at the joints must be the same as the total work at the hand frame.**

$$\delta W=\begin{bmatrix} f_x & f_y & f_z & m_x & m_y & m_z \end{bmatrix}\begin{bmatrix} dx \\ dy \\ dz \\ \partial x \\ \partial y \\ \partial z \end{bmatrix}= f_x dx+\cdots\cdots+m_z\delta z$$

$$\left[T\right]=\left[{}^{H}J\right]^T\left[{}^{H}F\right]$$

- **Referring to Appendix A**

TRANSFORMATION OF FORCES AND MOMENTS BETWEEN COORDINATE FRAMES

- **An equivalent force and moment with respect to the other coordinate frame by the principle of virtual work.**

$$[F]^T = [f_x \;\; f_y \;\; f_z \;\; m_x \;\; m_y \;\; m_z] \qquad [^BF]^T = [^Bf_x \;\; ^Bf_y \;\; ^Bf_z \;\; ^Bm_x \;\; ^Bm_y \;\; ^Bm_z]$$

$$[D]^T = [d_x \;\; d_y \;\; d_z \;\; \delta_x \;\; \delta_y \;\; \delta_z] \qquad [^BD]^T = [^Bd_x \;\; ^Bd_y \;\; ^Bd_z \;\; ^B\delta_x \;\; ^B\delta_y \;\; ^B\delta_z]$$

- **The total virtual work performed on the object in either frame must be the same.**

$$\delta W = [F]^T [D] = [^BT]^T [^BD]$$

TRANSFORMATION OF FORCES AND MOMENTS BETWEEN COORDINATE FRAMES

- **Displacements relative to the two frames are related to each other by the following relationship.**

$$\left[{}^{B}D\right]=\left[{}^{B}J\right][D]$$

- **The forces and moments with respect to frame *B* is can be calculated directly from the following equations:**

$$ {}^{B}f_x=\bar{n}\cdot\bar{f} \qquad {}^{B}m_x=\bar{n}\cdot[(\bar{f}\times\bar{p})+\bar{m}] $$

$$ {}^{B}f_y=\bar{o}\cdot\bar{f} \qquad {}^{B}m_y=\bar{o}\cdot[(\bar{f}\times\bar{p})+\bar{m}] $$

$$ {}^{B}f_y=\bar{o}\cdot\bar{f} \qquad {}^{B}m_z=\bar{a}\cdot[(\bar{f}\times\bar{p})+\bar{m}] $$

Thank you!